CPE2613: Signals and Systems

Fall 2026

Introduction

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Course Information

Reference Material

Lecture Slides

Class Handouts

Course Type	**Core Course**
Credit Hours	**3 Credit Hours**
Classes	**Three Lectures of 50 mins duration in a Week**
Class Timings	**1st and 2nd Wednesday, 14:20 to 16:10, 3rd : Thursday, 16:20 to 17:10**
Textbook	**Alan V. Oppenheim , "Signals and Systems"**
Ref Books	**1. Charles L. Phillip, "Signals, Systems, and Transforms"**
	2. B. P. Lathi, "Linear Systems and Signals"

Course Assessment

Quiz	30%	No. of Quizzes	5 (Best 4)
Assignments	10%	No. of Assignments	4
Midterm Exam	20%		
Final Exam	40%		



Re-Quiz or Re-Assignment

Class Rules



Use of Cell phone during lecture is not allowed. Otherwise, you will take cell phone from student affairs.

During quiz, all the students will keep their cell phones either in their bags, or in the side windows. ***Anyone caught with cell phone during quiz will get his quiz cancelled with 0 marks and the cell phone will be fined by student affairs.***

All discussions will be done before the assessments and exams. ***During any assessment like quiz, or exams, no questions or discussion are allowed.*** Even, asking for the explanation of question is not allowed, because, understanding the question is the part of question.

Any student facing problem related to course can either email or can take the office time for understanding and explanation.

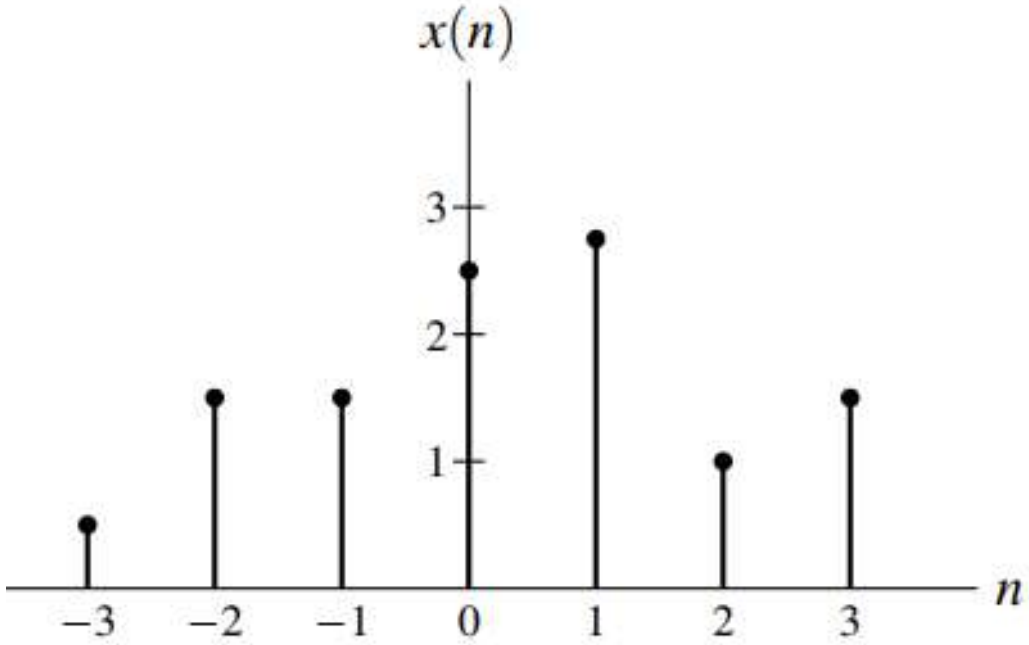
Signals

- A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- For a function f, in the expression $f(t_1, t_2, \ldots, t_n)$, each of the $\{t_k\}$ is called an **independent variable**, while the function value itself is referred to as a **dependent variable**.
- Some examples of signals include:
  - a voltage or current in an electronic circuit
  - the position, velocity, or acceleration of an object
  - a force or torque in a mechanical system
  - a flow rate of a liquid or gas in a chemical process
  - a digital image, digital video, or digital audio
  - a stock market index

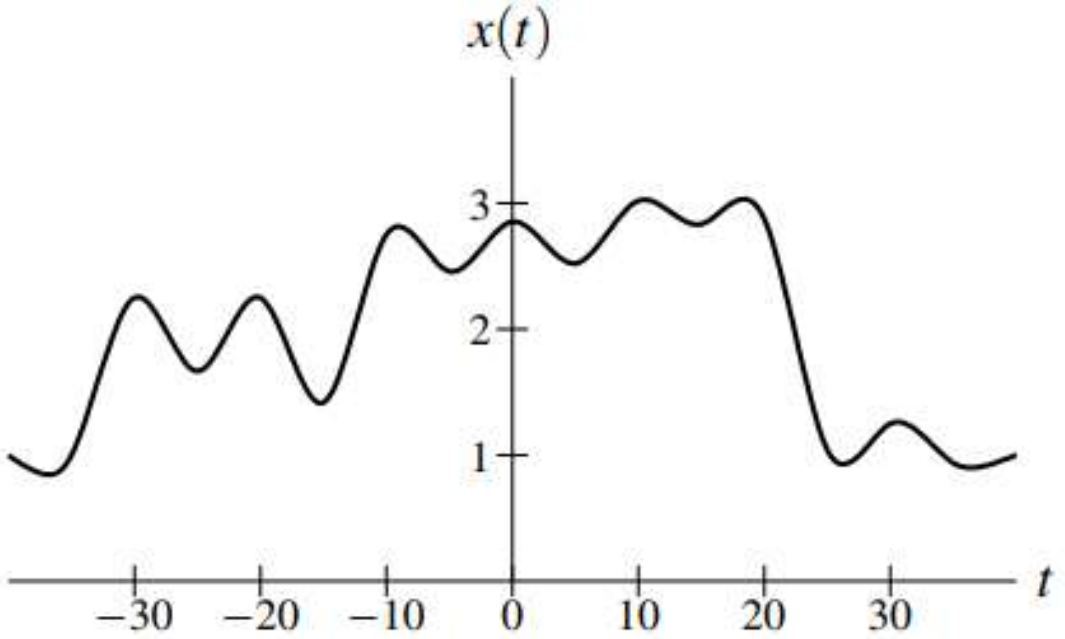
Classification of Signals

- Number of independent variables (i.e., dimensionality):
  - A signal with ***one*** independent variable is said to be **one dimensional** (e.g., audio).
  - A signal with ***more than one*** independent variable is said to be **multi-dimensional** (e.g., image).
- Continuous or discrete independent variables:
  - A signal with ***continuous*** independent variables is said to be **continuous time (CT)** (e.g., voltage waveform).
  - A signal with ***discrete*** independent variables is said to be **discrete time (DT)** (e.g., stock market index).
- Continuous or discrete dependent variable:
  - A signal with a ***continuous*** dependent variable is said to be **continuous valued** (e.g., voltage waveform).
  - A signal with a ***discrete*** dependent variable is said to be **discrete valued** (e.g., digital image).
- A ***continuous-valued CT*** signal is said to be **analog** (e.g., voltage waveform).
- A ***discrete-valued DT*** signal is said to be **digital** (e.g., digital audio).

Graphical Representation of Signals



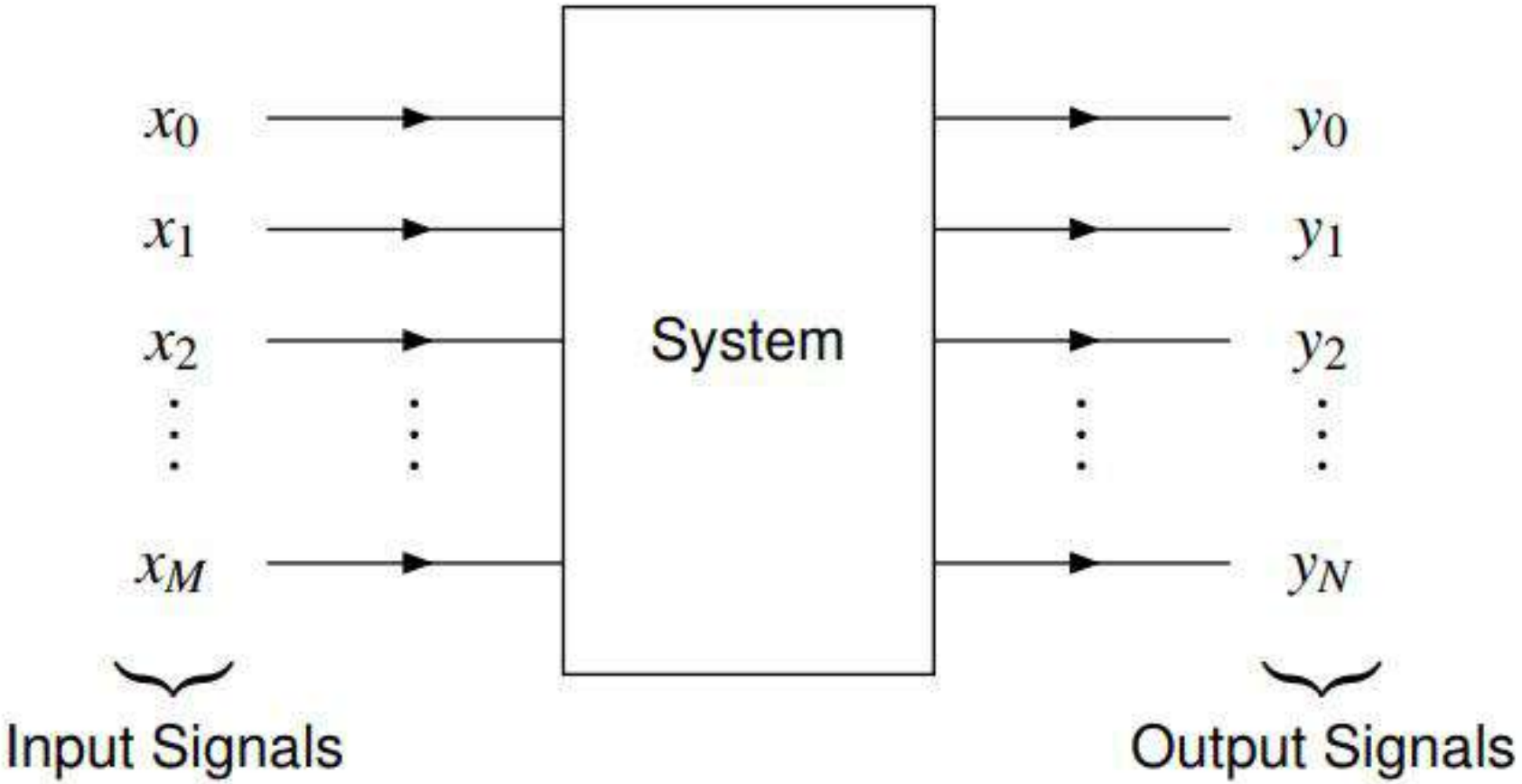
Discrete-Time (DT) Signal



Continuous-Time (CT) Signal

Systems

- A system is an entity that processes one or more input signals in order to produce one or more output signals.



Classification of Systems

- Number of inputs:
  - A system with ***one*** input is said to be **single input (SI)**.
  - A system with ***more than one*** input is said to be **multiple input (MI)**.
- Number of outputs:
  - A system with ***one*** output is said to be **single output (SO)**.
  - A system with ***more than one*** output is said to be **multiple output (MO)**.
- Types of signals processed:
  - A system can be classified in terms of the ***types of signals*** that it processes.
  - Consequently, terms such as the following (which describe signals) can also be used to describe systems:
    - one-dimensional and multi-dimensional,
    - continuous-time (CT) and discrete-time (DT), and
    - analog and digital.
  - For example, a continuous-time (CT) system processes CT signals and a discrete-time (DT) system processes DT signals.

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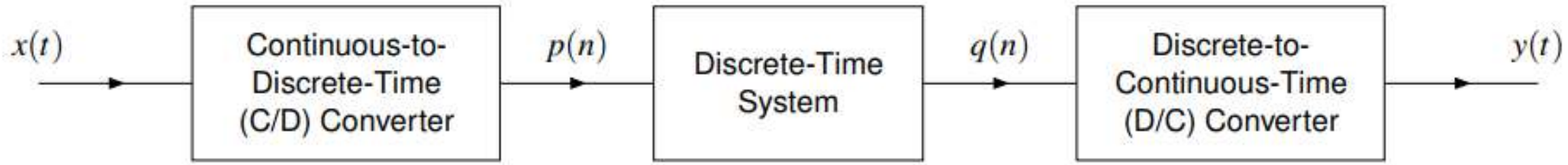
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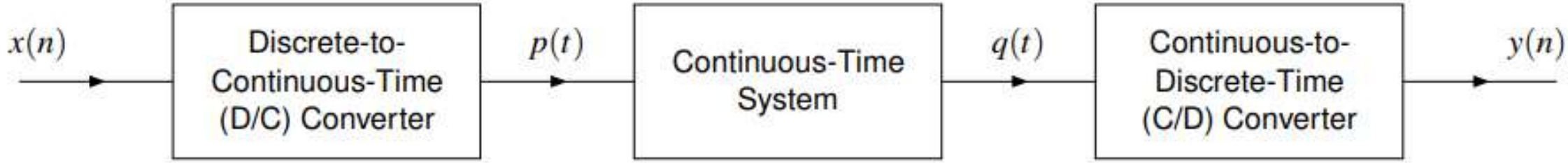
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Signal Processing Systems



Processing a Continuous-Time Signal With a Discrete-Time System

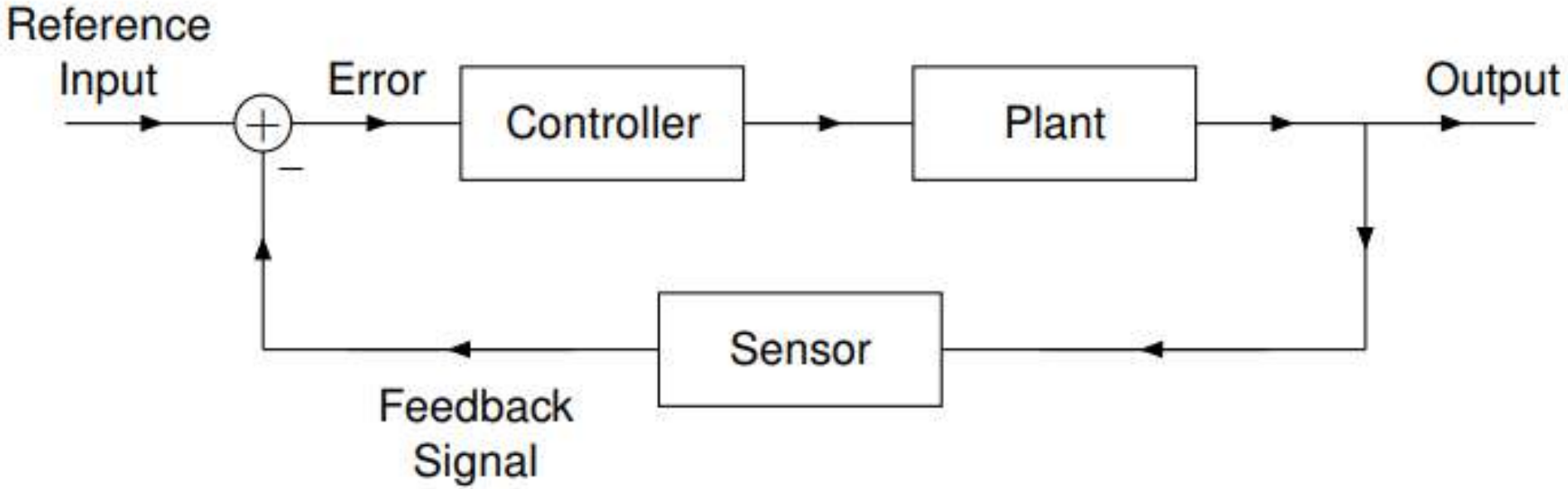


Processing a Discrete-Time Signal With a Continuous-Time System

Communication and Feedback Control Systems



General Structure of a Communication System



General Structure of a Feedback Control System

Why Study Signals and Systems?

- Engineers build systems that process/manipulate signals.
- We need a formal mathematical framework for the study of such systems.
- Such a framework is necessary in order to ensure that a system will meet the required specifications (e.g., performance and safety).
- If a system fails to meet the required specifications or fails to work altogether, negative consequences usually ensue.
- When a system fails to operate as expected, the consequences can sometimes be catastrophic.

Signals

- Earlier, we were introduced to CT and DT signals.
- A CT signal is called a **function**.
- A DT signal is called a **sequence**.
- Although, strictly speaking, a sequence is a special case of a function (where the domain of the function is the integers), we will use the term function exclusively to mean a function that is not a sequence.
- The nth element of a sequence x is denoted as either $x(n)$ or x_n.

Notation: Functions Versus Function Values

- Strictly speaking, an expression like "$f(t)$" means the ***value*** of the function f evaluated at the point t.
- Unfortunately, engineers often use an expression like "$f(t)$" to refer to the ***function*** f (rather than the value of f evaluated at the point t), and this sloppy notation can lead to problems (e.g., ambiguity) in some situations.
- In contexts where sloppy notation may lead to problems, one should be careful to clearly distinguish between a function and its value.
- Example (meaning of notation):
  - Let f and g denote real-valued functions of a real variable.
  - Let t denote an arbitrary real number.
  - Let $\mathcal{H}$ denote a system operator (which maps a function to a function).
  - The quantity $f+g$ is a ***function***, namely, the function formed by adding the functions f and g.
  - The quantity $f(t)+g(t)$ is a ***number***, namely, the sum of: the value of the function f evaluated at t; and the value of the function g evaluated at t.
  - The quantity $\mathcal{H}x$ is a ***function***, namely, the output produced by the system represented by $\mathcal{H}$ when the input to the system is the function x.
  - The quantity $\mathcal{H}x(t)$ is a ***number***, namely, the value of the function $\mathcal{H}x$ evaluated at t.

Properties of Signals

Even Signals

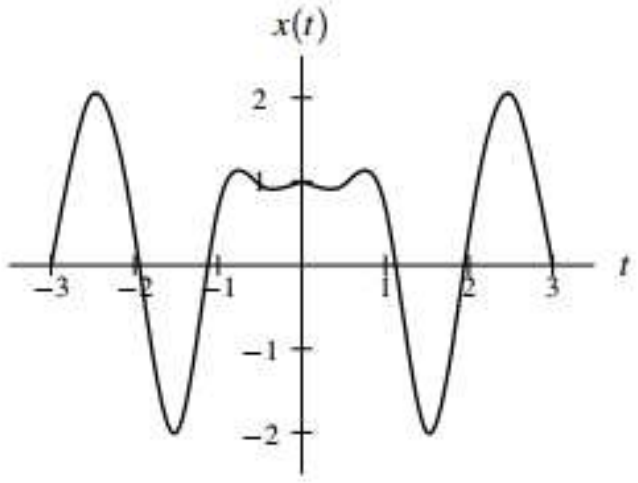
- A function x is said to be **even** if it satisfies

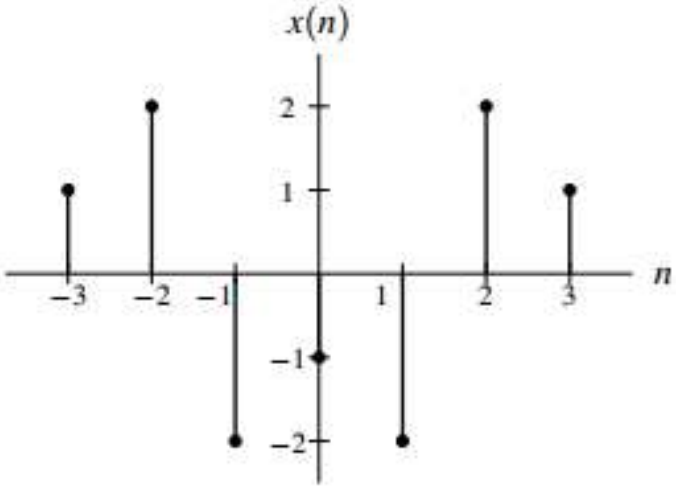
$$x(t) = x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an even signal is *symmetric* about the origin.
- Some examples of even signals are shown below.





Properties of Signals

Odd Signals

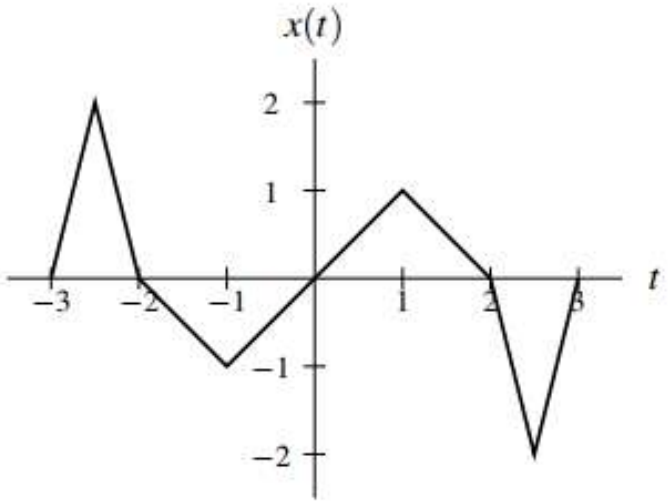
- A function x is said to be **odd** if it satisfies

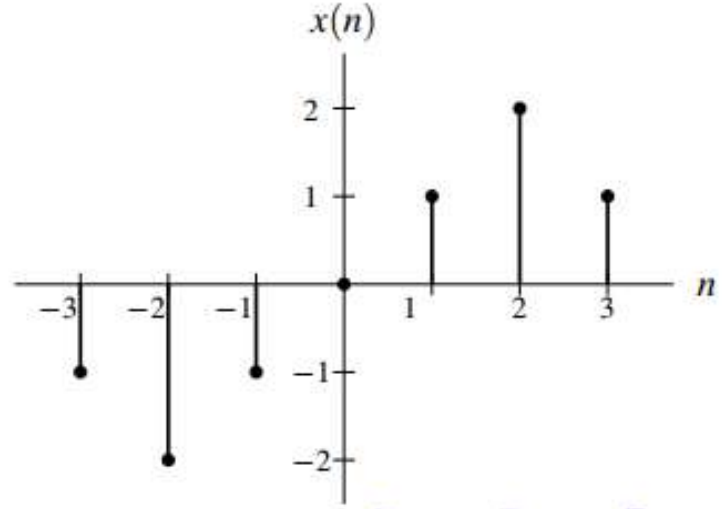
$$x(t) = -x(-t) \quad \text{for all } t.$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n.$$

- Geometrically, the graph of an odd signal is ***antisymmetric*** about the origin.
- An odd signal x must be such that $x(0) = 0$.
- Some examples of odd signals are shown below.





Symmetry and Addition/Multiplication

- Sums involving even and odd functions have the following properties:
  - The sum of two even functions is even.
  - The sum of two odd functions is odd.
  - The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.
- That is, the ***sum*** of functions with the ***same type of symmetry*** also has the ***same type of symmetry***.
- Products involving even and odd functions have the following properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
- That is, the ***product*** of functions with the ***same type of symmetry*** is ***even***, while the ***product*** of functions with ***opposite types of symmetry*** is ***odd***.

Decomposition of a Signal into Even and Odd Parts

- Every function x has a ***unique*** representation of the form

$$x(t) = x_{\mathrm{e}}(t) + x_{\mathrm{o}}(t),$$

  where the functions x_{e} and x_{o} are ***even*** and ***odd***, respectively.

- In particular, the functions x_{e} and x_{o} are given by

$$x_{\mathrm{e}}(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_{\mathrm{o}}(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

- The functions x_{e} and x_{o} are called the **even part** and **odd part** of x, respectively.

- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

Properties of Signals

Periodic Signals

- A function x is said to be **periodic** with **period** T (or **T-periodic**) if, for some strictly-positive real constant T, the following condition holds:

$$x(t) = x(t+T) \quad \text{for all } t.$$

- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- A sequence x is said to be **periodic** with **period** N (or **N-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

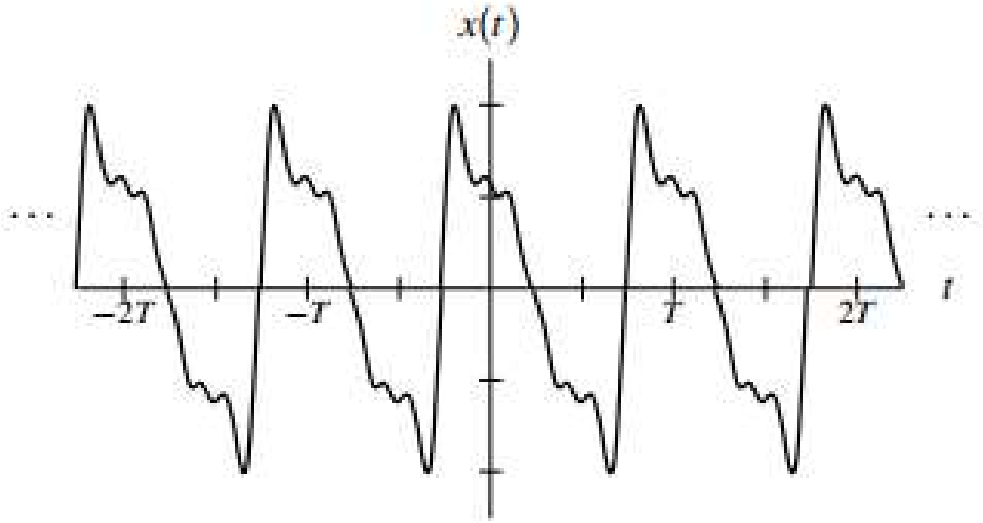
$$x(n) = x(n+N) \quad \text{for all } n.$$

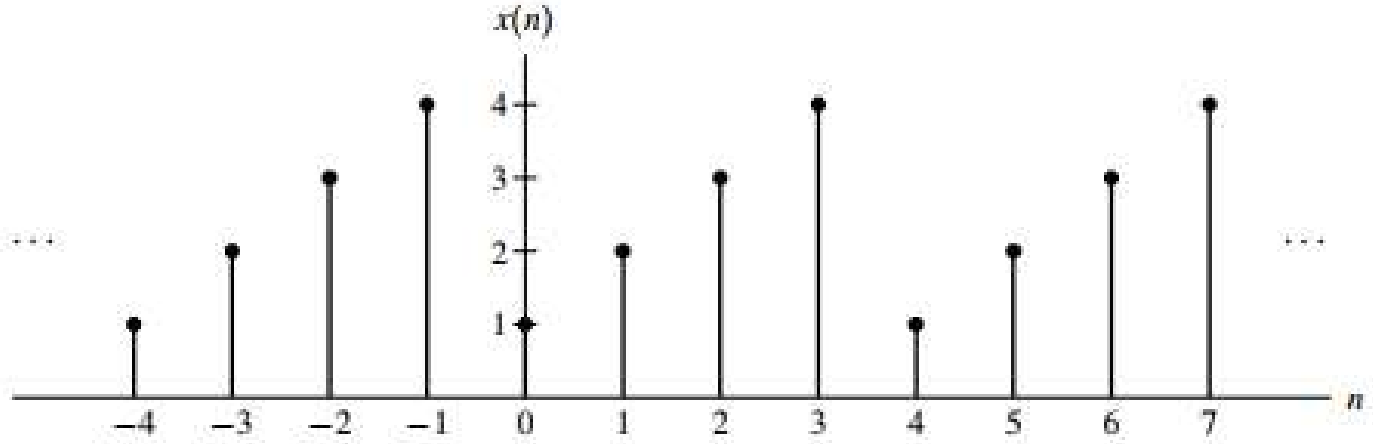
- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.
- A function/sequence that is not periodic is said to be **aperiodic**.

Properties of Signals

Periodic Signals

- Some examples of periodic signals are shown below.

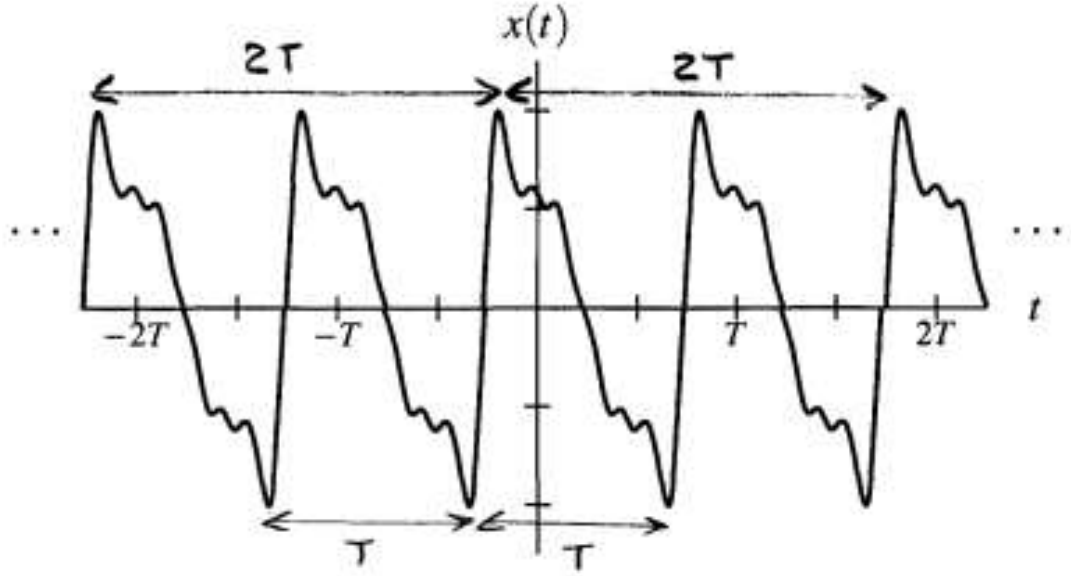




Properties of Signals

Periodic Signals

- The period of a periodic signal is ***not unique***. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.



- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Sum of Periodic Functions

- **Sum of periodic functions.** Let x_1 and x_2 be periodic functions with fundamental periods T_1 and T_2, respectively. Then, the sum $y = x_1 + x_2$ is a periodic function if and only if the ratio T_1/T_2 ***is a rational number*** (i.e., the quotient of two integers). Suppose that $T_1/T_2 = q/r$ where q and r are integers and ***coprime*** (i.e., have no common factors), then the fundamental period of y is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)
- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N - 1$ times.

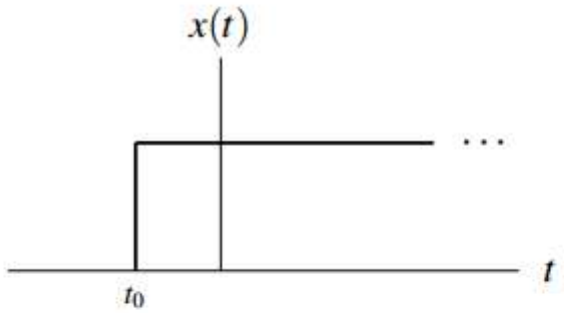
Right-Sided Signals

- A signal x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e., x is ***only potentially nonzero to the right of*** t_0).

- An example of a right-sided signal is shown below.



- A signal x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal signal is a ***special case*** of a right-sided signal.
- A causal signal is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

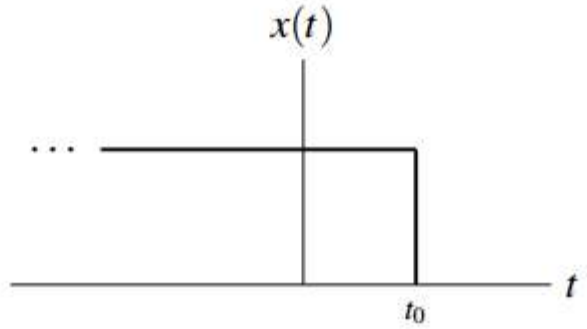
Left-Sided Signals

- A signal x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is ***only potentially nonzero to the left of*** t_0).

- An example of a left-sided signal is shown below.



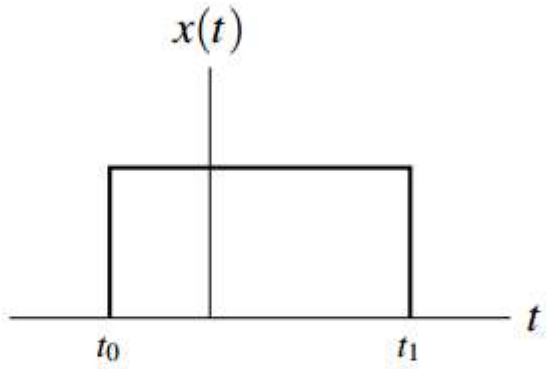
- Similarly, a signal x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

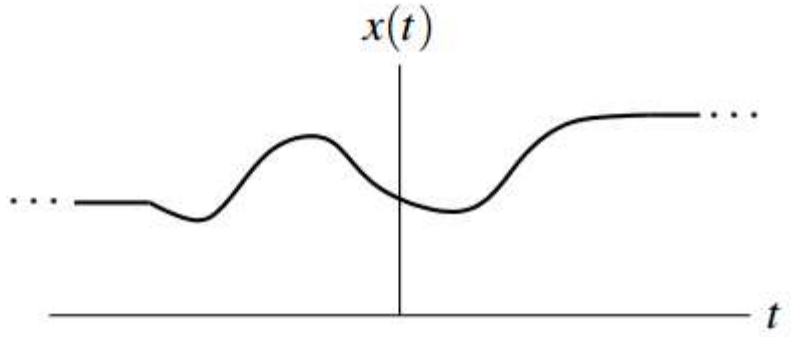
- An anticausal signal is a ***special case*** of a left-sided signal.
- An anticausal signal is not to be confused with an anticausal system. In these two contexts, the word "anticausal" has very different meanings.

Finite-Duration and Two-Sided Signals

- A signal that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration signal is shown below.



- A signal that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided signal is shown below.



Bounded Signals

Lecture 3

- A signal x is said to be **bounded** if there exists some (***finite***) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is ***finite*** for all t).

- Examples of bounded signals include the sine and cosine functions.
- Examples of unbounded signals include the tan function and any nonconstant polynomial function.

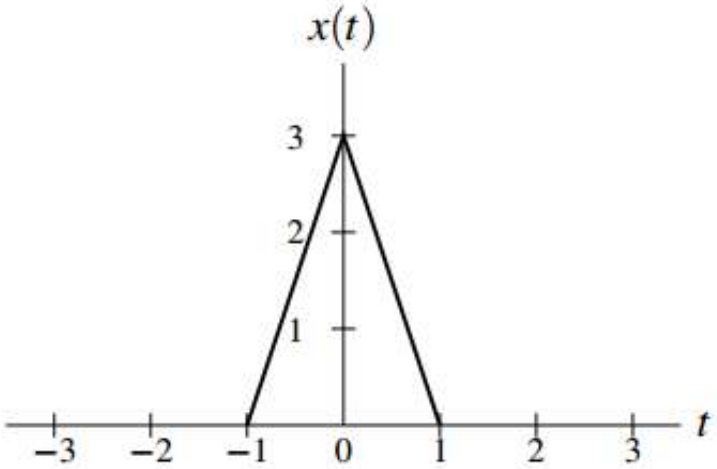
Time Shifting (Translation)

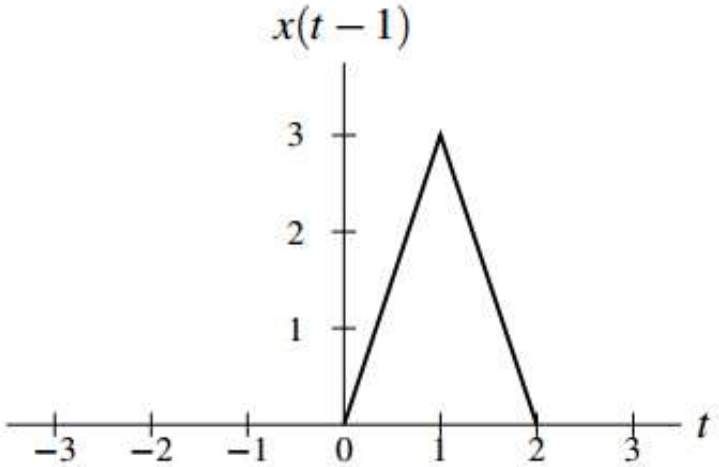
- **Time shifting** (also called **translation**) maps the input signal x to the output signal y as given by

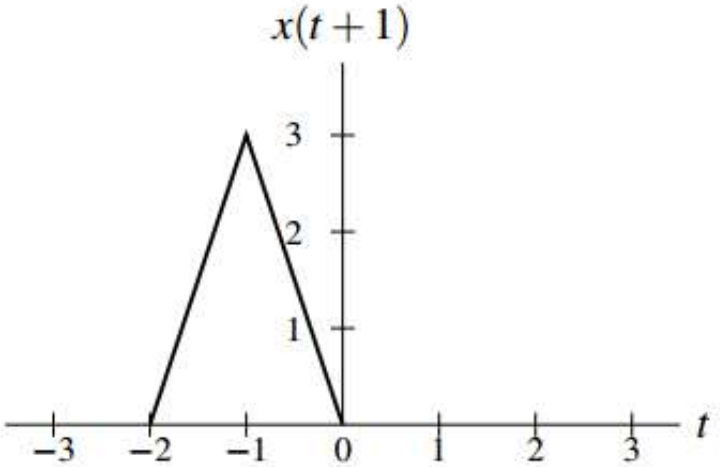
$$y(t) = x(t - b),$$

  where b is a real number.
- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

Time Shifting (Translation): Example





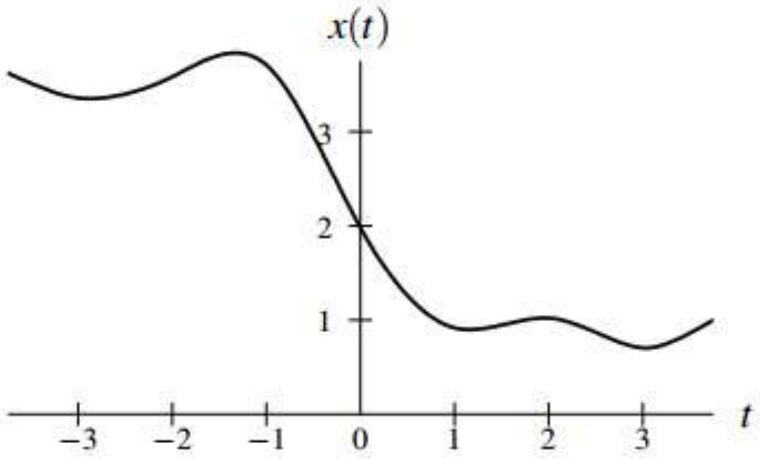


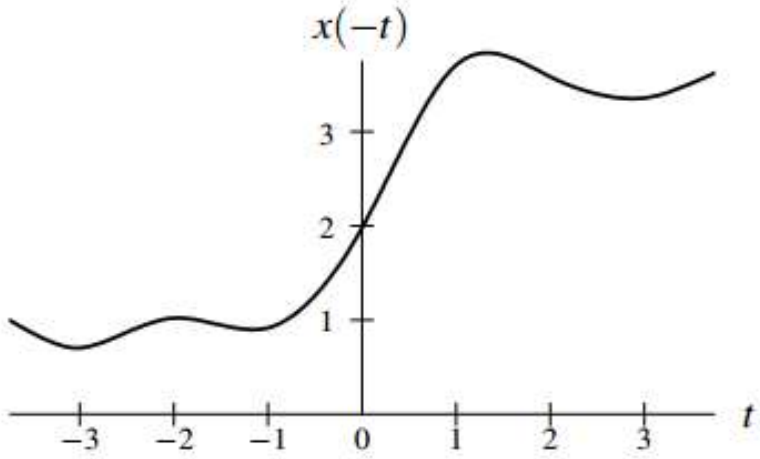
Time Reversal (Reflection)

- **Time reversal** (also known as **reflection**) maps the input signal x to the output signal y as given by

$$y(t) = x(-t).$$

- Geometrically, the output signal y is a reflection of the input signal x about the (vertical) line $t = 0$.





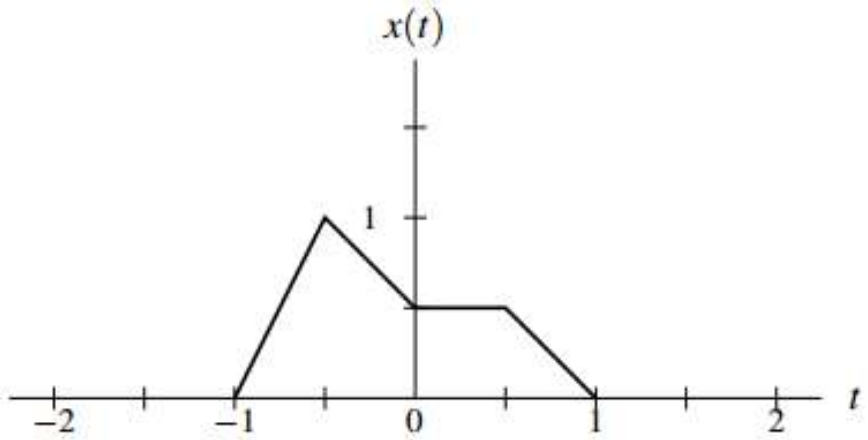
Time Compression/Expansion (Dilation)

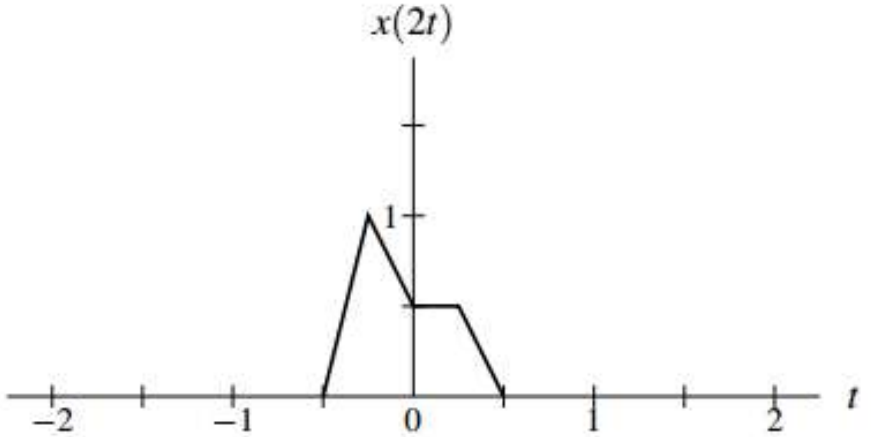
- **Time compression/expansion** (also called **dilation**) maps the input signal x to the output signal y as given by

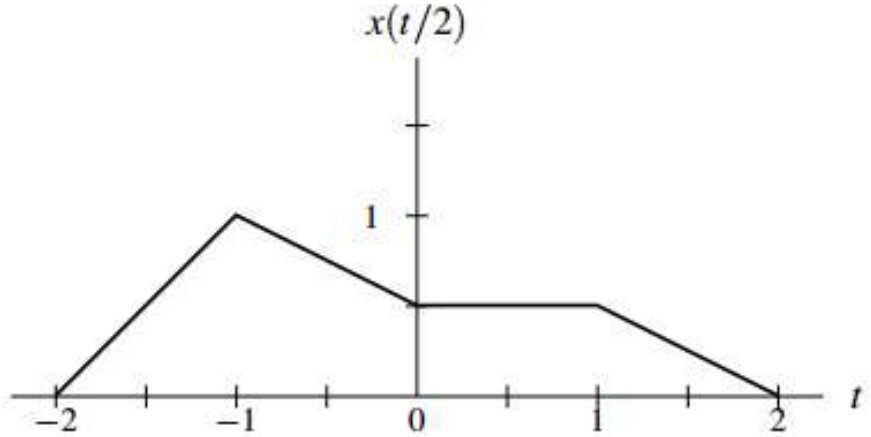
$$y(t) = x(at),$$

  where a is a ***strictly positive*** real number.
- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, y is ***compressed*** along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, y is ***expanded*** (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

Time Compression/Expansion (Dilation): Example







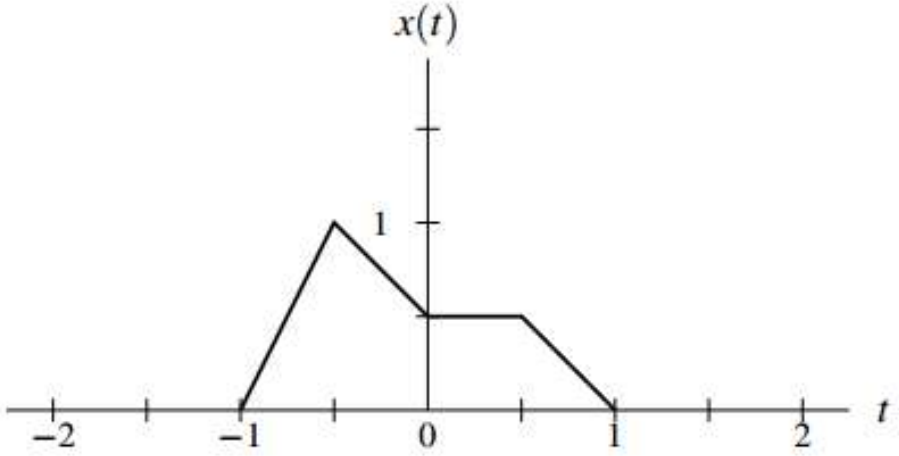
Time Scaling

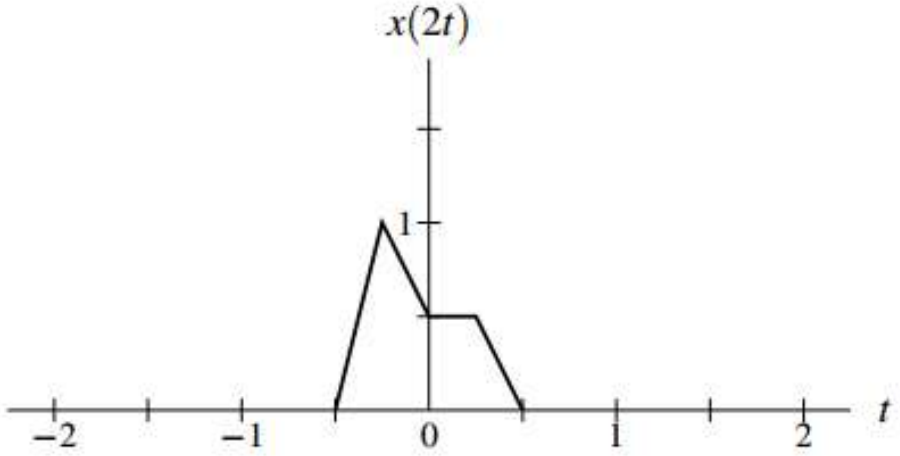
- **Time scaling** maps the input signal x to the output signal y as given by

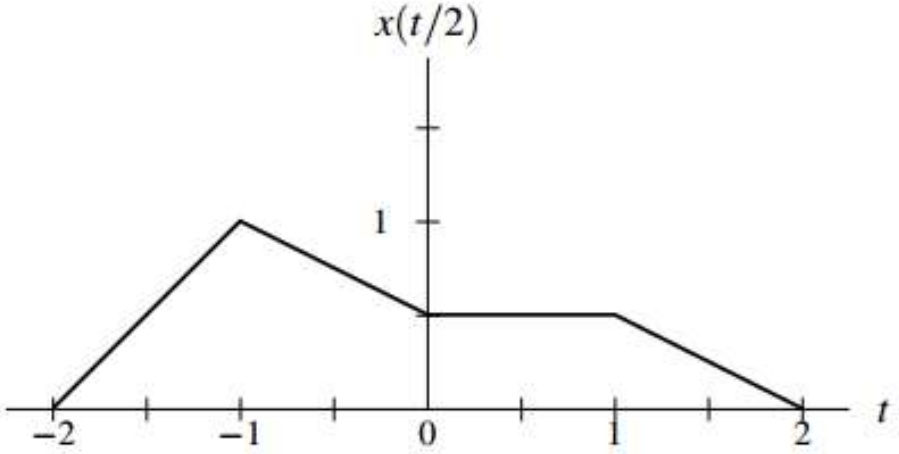
$$y(t) = x(at),$$

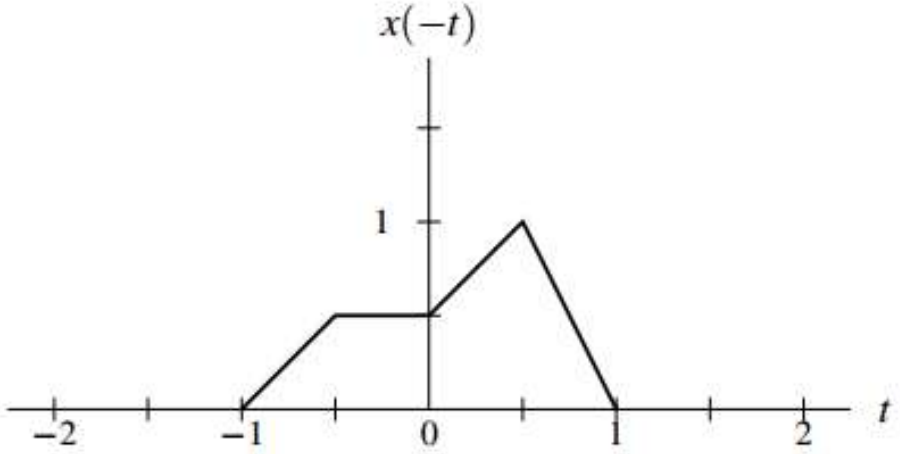
  where a is a ***nonzero*** real number.
- Such a transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or time reversal.
- If $|a| > 1$, the signal is ***compressed*** along the time axis by a factor of $|a|$.
- If $|a| < 1$, the signal is ***expanded*** (i.e., stretched) along the time axis by a factor of $\left|\frac{1}{a}\right|$.
- If $|a| = 1$, the signal is neither expanded nor compressed.
- If $a < 0$, the signal is also time reversed.
- Dilation (i.e., expansion/compression) and time reversal ***commute***.
- Time reversal is a special case of time scaling with $a = -1$; and time compression/expansion is a special case of time scaling with $a > 0$.

Time Scaling (Dilation/Reflection): Example









Determine whether or not each of the following signals is periodic:

(a) $x_1(t) = 2e^{j(t+\pi/4)}u(t)$ **(b)** $x_2[n] = u[n] + u[-n]$

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}\{\delta[n-4k] - \delta[n-1-4k]\}$

Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(a) $x_1(t) = je^{j10t}$ **(b)** $x_2(t) = e^{(-1+j)t}$ **(c)** $x_3[n] = e^{j7\pi n}$

(d) $x_4[n] = 3e^{j3\pi(n+1/2)/5}$ **(e)** $x_5[n] = 3e^{j3/5(n+1/2)}$

Determine the fundamental period of the signal $x(t) = 2\cos(10t+1) - \sin(4t-1)$.

Determine the fundamental period of the signal $x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$.

A continuous-time signal $x(t)$ is shown in Figure P1.21. Sketch and label carefully each of the following signals:

(a) $x(t-1)$ **(b)** $x(2-t)$ **(c)** $x(2t+1)$

(d) $x(4-\frac{t}{2})$

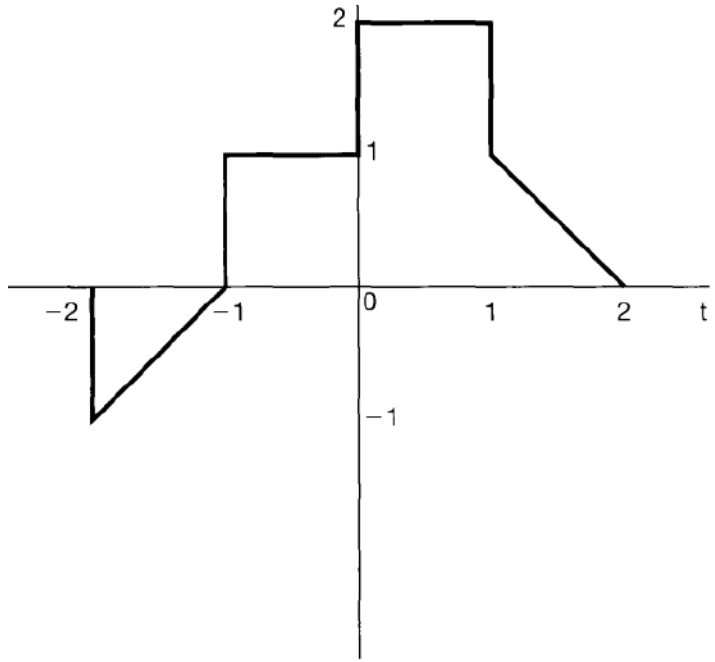


Figure P1.21

A discrete-time signal is shown in Figure P1.22. Sketch and label carefully each of the following signals:

(a) $x[n-4]$ **(b)** $x[3-n]$ **(c)** $x[3n]$

(d) $x[3n+1]$

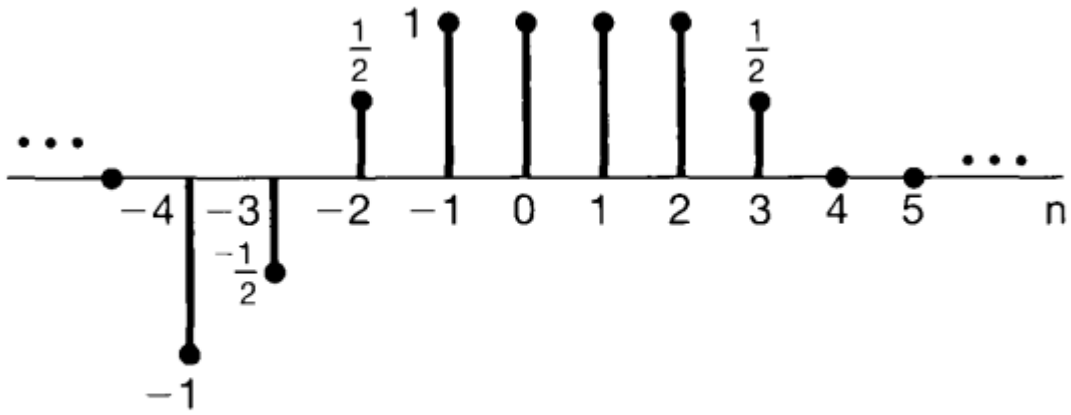


Figure P1.22

Determine whether or not each of the following continuous-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 3\cos(4t + \frac{\pi}{3})$ **(b)** $x(t) = e^{j(\pi t - 1)}$

(c) $x(t) = [\cos(2t - \frac{\pi}{3})]^2$

Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = \sin(\frac{6\pi}{7}n + 1)$ **(b)** $x[n] = \cos(\frac{n}{8} - \pi)$ **(c)** $x[n] = \cos(\frac{\pi}{8}n^2)$

(d) $x[n] = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{4}n)$

Let $x(t)$ be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal $y_1(t)$ represents a speeded up version of $x(t)$ in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of $x(t)$ in the sense that the duration of the signal is doubled. Consider the following statements:

(1) If $x(t)$ is periodic, then $y_1(t)$ is periodic.
(2) If $y_1(t)$ is periodic, then $x(t)$ is periodic.
(3) If $x(t)$ is periodic, then $y_2(t)$ is periodic.
(4) If $y_2(t)$ is periodic, then $x(t)$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

Let $x[n]$ be a discrete-time signal, and let

$$y_1[n] = x[2n] \text{ and } y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

The signals $y_1[n]$ and $y_2[n]$ respectively represent in some sense the speeded up and slowed down versions of $x[n]$. However, it should be noted that the discrete-time notions of speeded up and slowed down have subtle differences with respect to their continuous-time counterparts. Consider the following statements:

(1) If $x[n]$ is periodic, then $y_1[n]$ is periodic.
(2) If $y_1[n]$ is periodic, then $x[n]$ is periodic.
(3) If $x[n]$ is periodic, then $y_2[n]$ is periodic.
(4) If $y_2[n]$ is periodic, then $x[n]$ is periodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. If the statement is not true, produce a counterexample to it.

Combined Time Scaling and Time Shifting

- Consider a transformation that maps the input signal x to the output signal y as given by

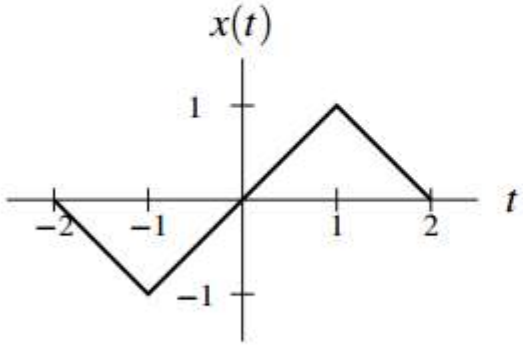
$$y(t) = x(at - b),$$

where a and b are real numbers and $a \neq 0$.

- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting ***do not commute***, we must be particularly careful about the order in which these transformations are applied.
- The above transformation has two distinct but equivalent interpretations:
  1. first, time shifting x by b, and then time scaling the result by a;
  2. first, time scaling x by a, and then time shifting the result by b/a.
- Note that the time shift is not by the same amount in both cases.

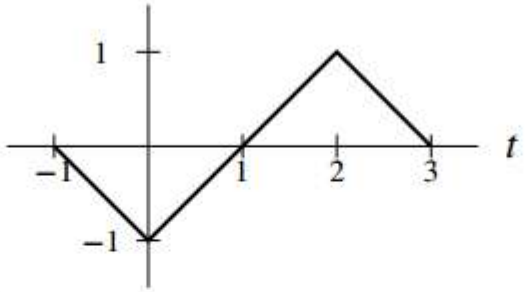
Combined Time Scaling and Time Shifting: Example

Given $x(t)$ as shown below, find $x(2t-1)$.

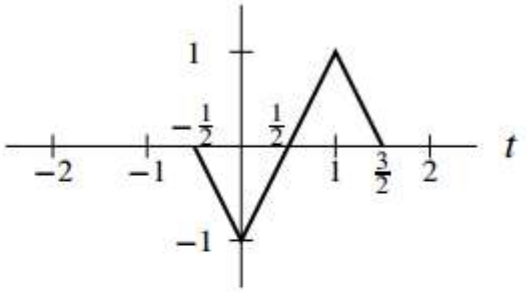


time shift by 1 and then time scale by 2

$p(t) = x(t-1)$

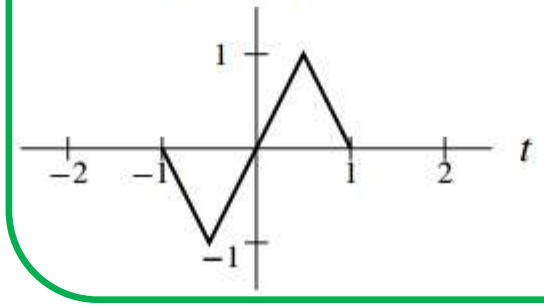


$p(2t) = x(2t-1)$

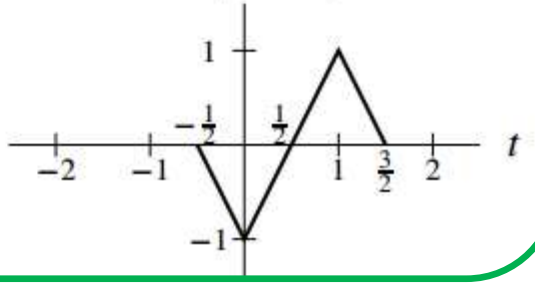


time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$



$q(t-1/2) = x(2(t-1/2)$
$= x(2t-1)$



Two Perspectives on Independent-Variable Transformations

- A transformation of the independent variable can be viewed in terms of
  1. the effect that the transformation has on the ***signal***; or
  2. the effect that the transformation has on the ***horizontal axis***.
- This distinction is important because such a transformation has ***opposite*** effects on the signal and horizontal axis.
- For example, the (time-shifting) transformation that replaces t by $t - b$ (where b is a real number) in $x(t)$ can be viewed as a transformation that
  1. shifts the signal x ***right*** by b units; or
  2. shifts the horizontal axis ***left*** by b units.
- In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the ***signal***.
- If one is not careful to consider that we are interested in the signal perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.

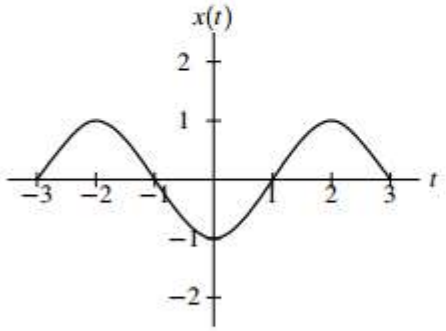
Amplitude Scaling

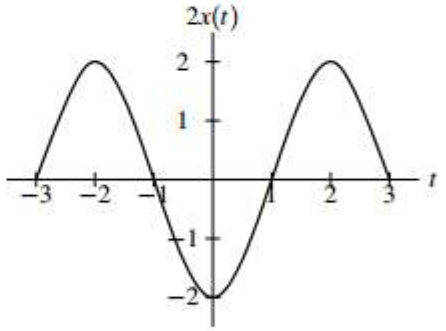
- **Amplitude scaling** maps the input signal x to the output signal y as given by

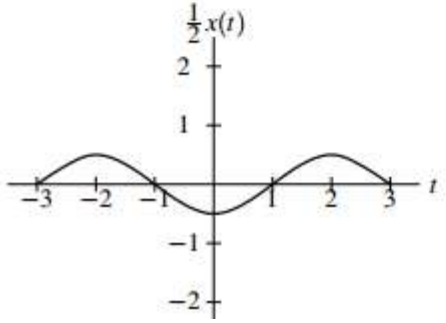
$$y(t) = ax(t),$$

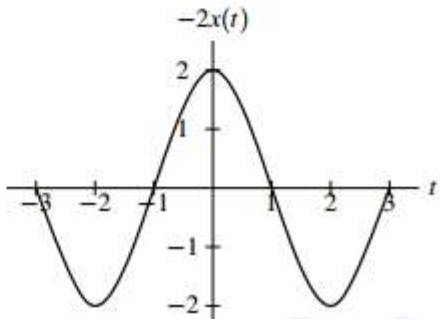
where a is a real number.

- Geometrically, the output signal y is ***expanded/compressed*** in amplitude and/or ***reflected*** about the horizontal axis.









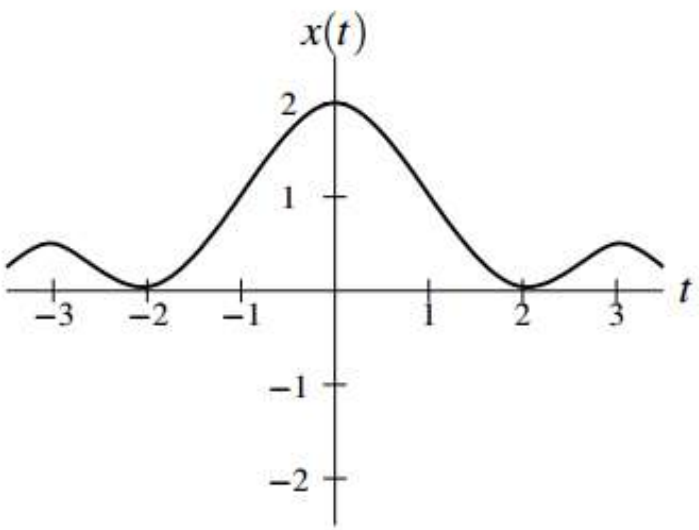
Amplitude Shifting

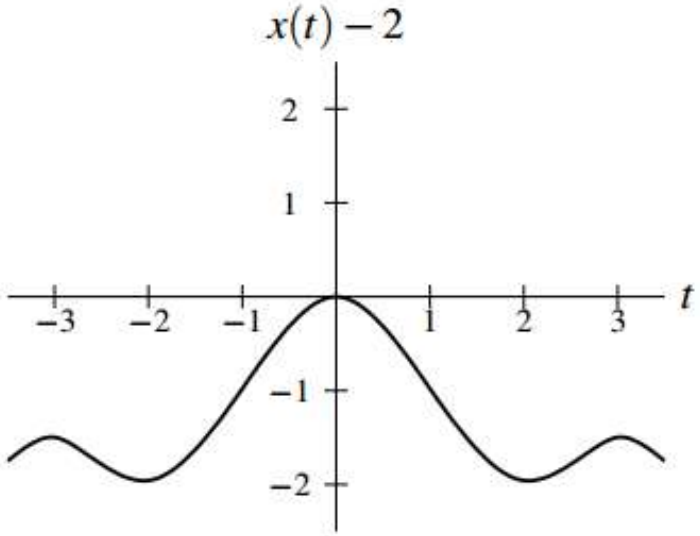
- **Amplitude shifting** maps the input signal x to the output signal y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to x.





Combined Amplitude Scaling and Amplitude Shifting

- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input signal x to the output signal y, as given by

$$y(t) = ax(t) + b,$$

where a and b are real numbers.

- Equivalently, the above transformation can be expressed as

$$y(t) = a\left[x(t) + \frac{b}{a}\right].$$

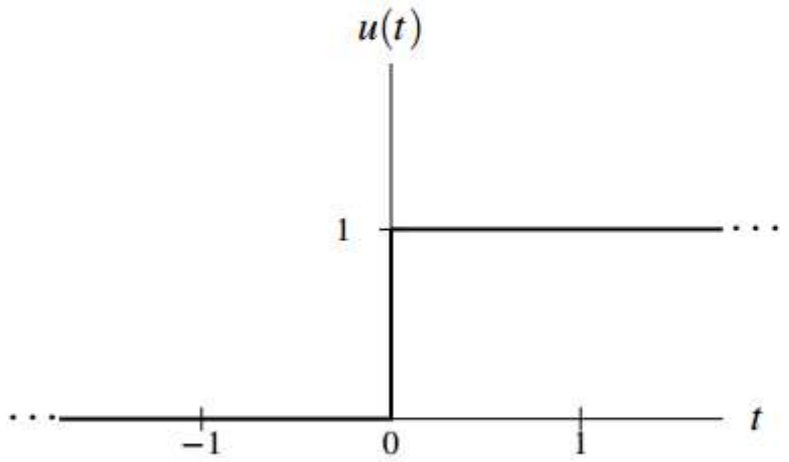
- The above transformation is equivalent to:
  1. first amplitude scaling x by a, and then amplitude shifting the resulting signal by b; or
  2. first amplitude shifting x by b/a, and then amplitude scaling the resulting signal by a.

Unit-Step Function

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which u is used in practice, the actual ***value of*** $u(0)$ is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.

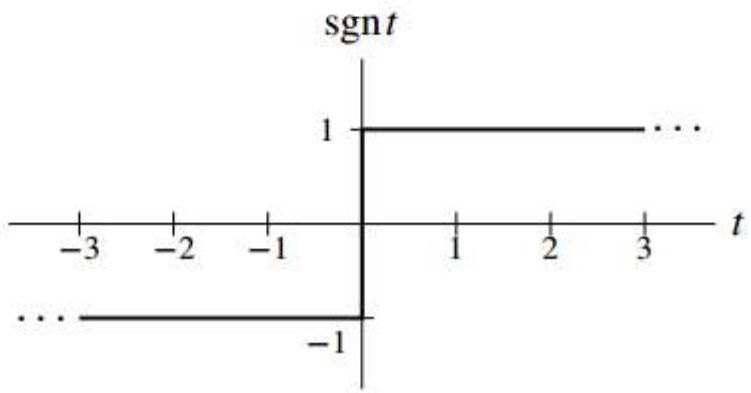


Signum Function

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the ***sign*** of a number.
- A plot of this function is shown below.

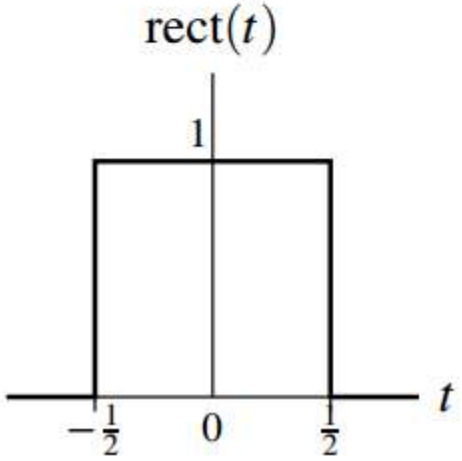


Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\text{rect}(t)$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.

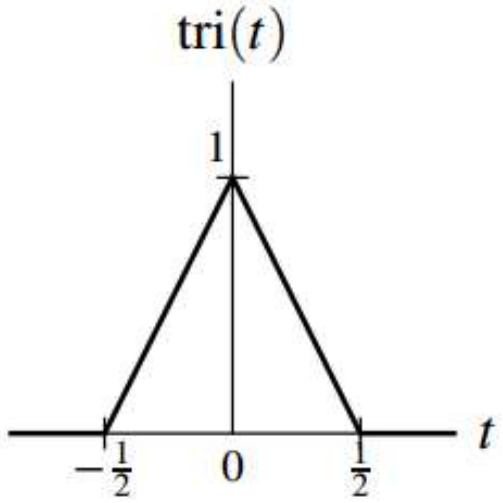


Triangular Function

- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\text{tri}(t) = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this function is shown below.

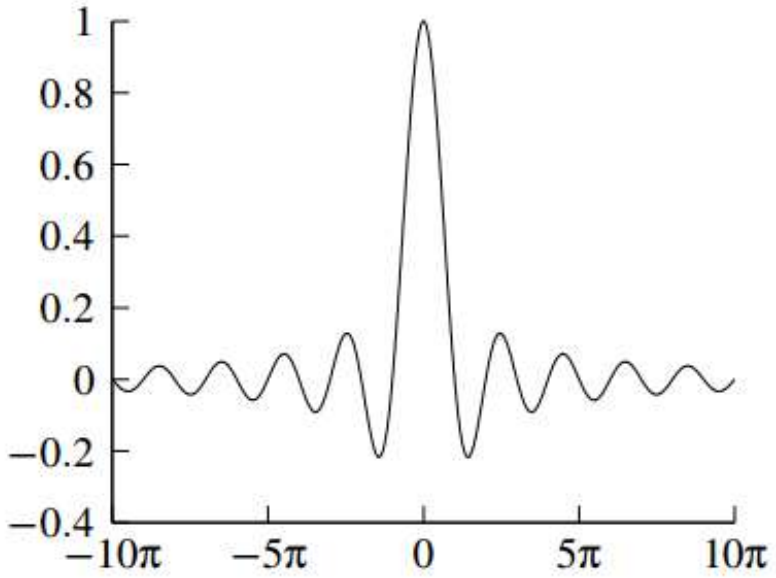


Cardinal Sine Function

- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc}(t) = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc}(t)$ do not die out for finite t.]



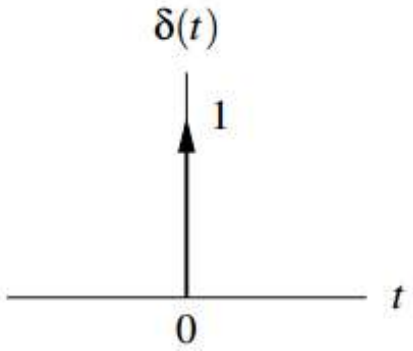
Unit-Impulse Function

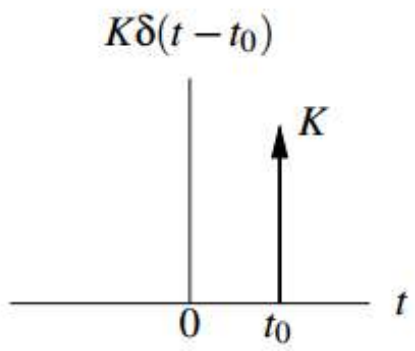
- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a ***generalized function***. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.



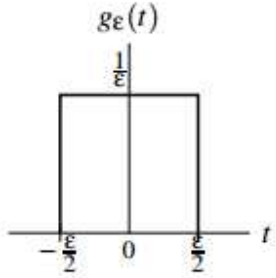


Unit-Impulse Function as a Limit

- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

- That is, δ can be viewed as a ***limiting case of a rectangular pulse*** where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Properties of the Unit-Impulse Function

- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$

- The δ function also has the following properties:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \frac{1}{|a|}\delta(t),$$

where a is a nonzero real constant.

Representing a Rectangular Pulse Using Unit-Step Functions

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

  (i.e., $x(t)$ is a ***rectangular pulse*** of height one, with a ***rising edge at*** a and ***falling edge at*** b).

- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

  (i.e., the difference of two time-shifted unit-step functions).

- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.

- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

Representing Functions Using Unit-Step Functions

- The idea from the previous slide can be extended to handle any function that is defined in a ***piecewise manner*** (i.e., via an expression involving multiple cases).
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Signal Energy and Power

- The **energy** E contained in the signal x is given by

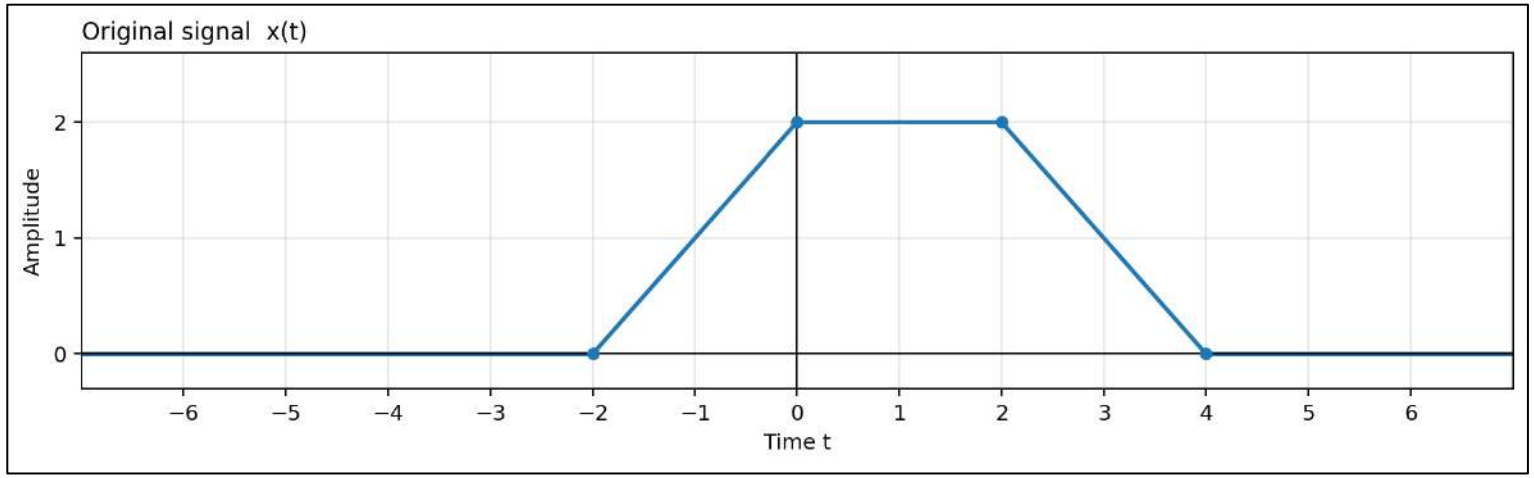
$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the signal x is given by

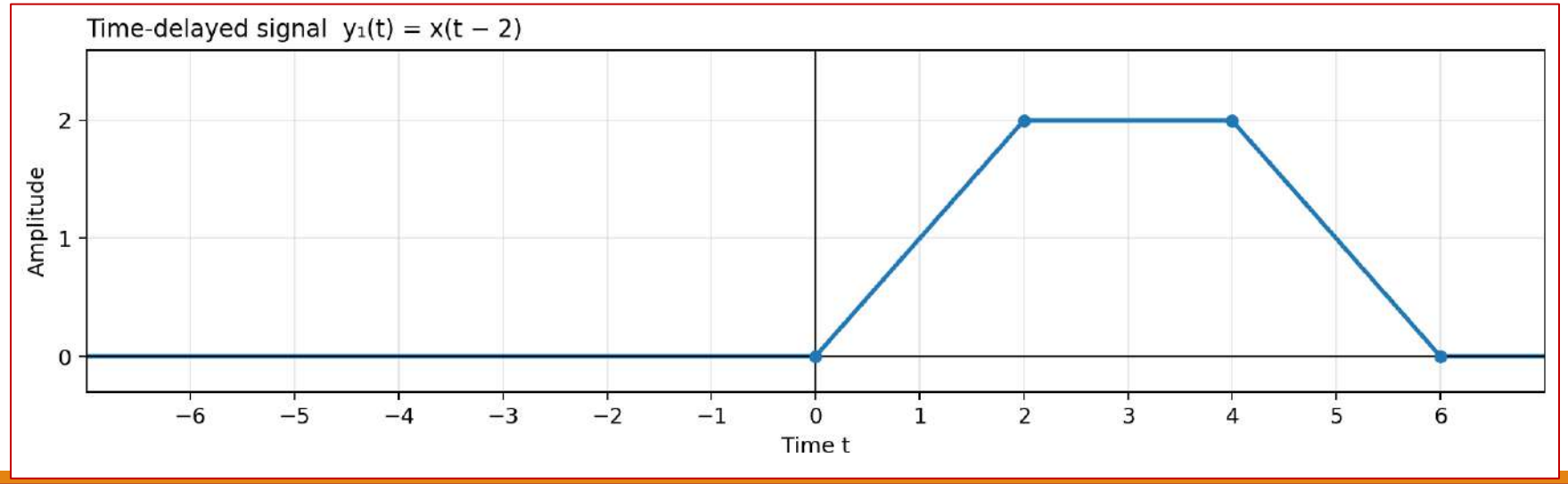
$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

- A signal with (nonzero) finite average power is said to be a **power signal**.

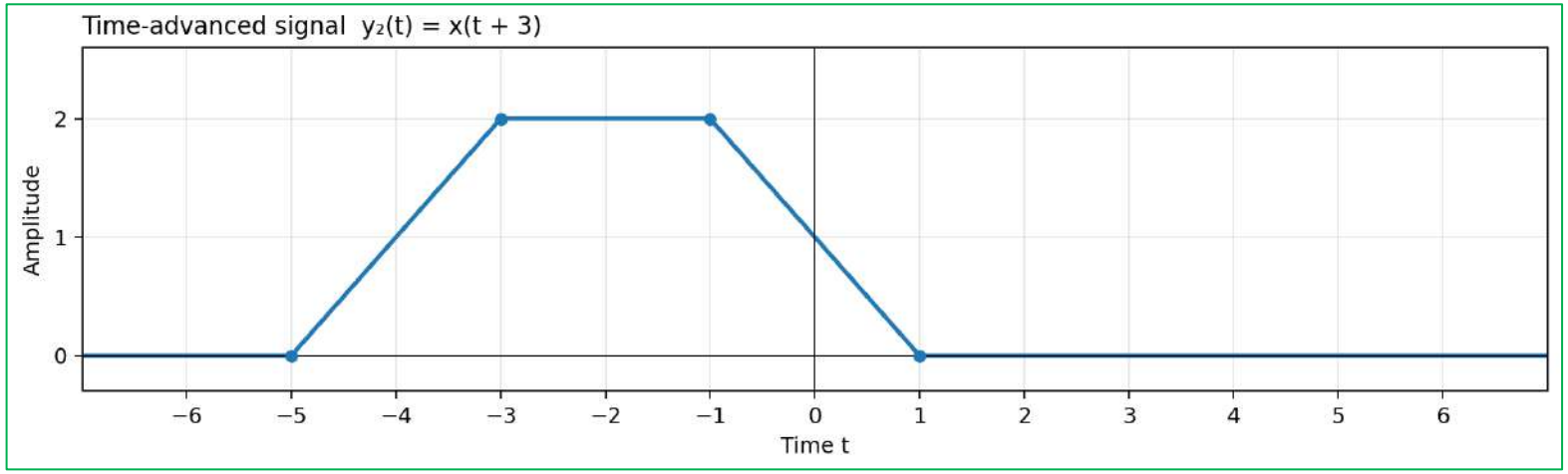
$$x(t) = \begin{cases} t+2, & -2 \le t < 0 \\ 2, & 0 \le t < 2 \\ 4-t, & 2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$



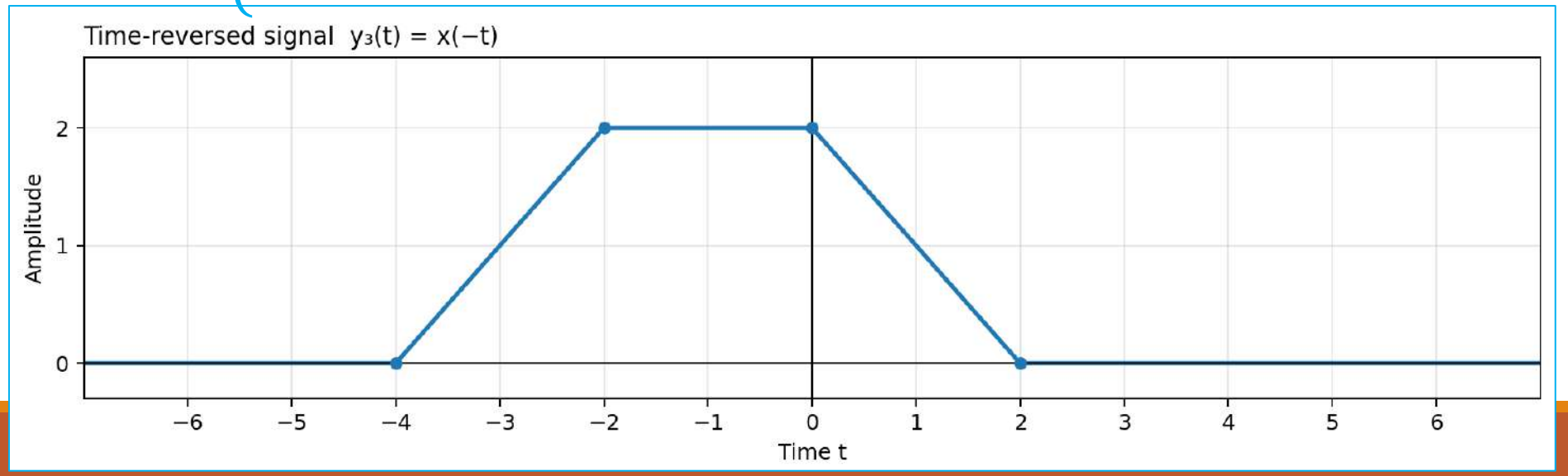
$$y_1(t) = \mathrm{x}(\mathrm{t}-2) \begin{cases} t, & 0 \le t < 2 \\ 2, & 2 \le t < 4 \\ 6-t, & 4 \le t \le 6 \\ 0, & \text{otherwise} \end{cases}$$



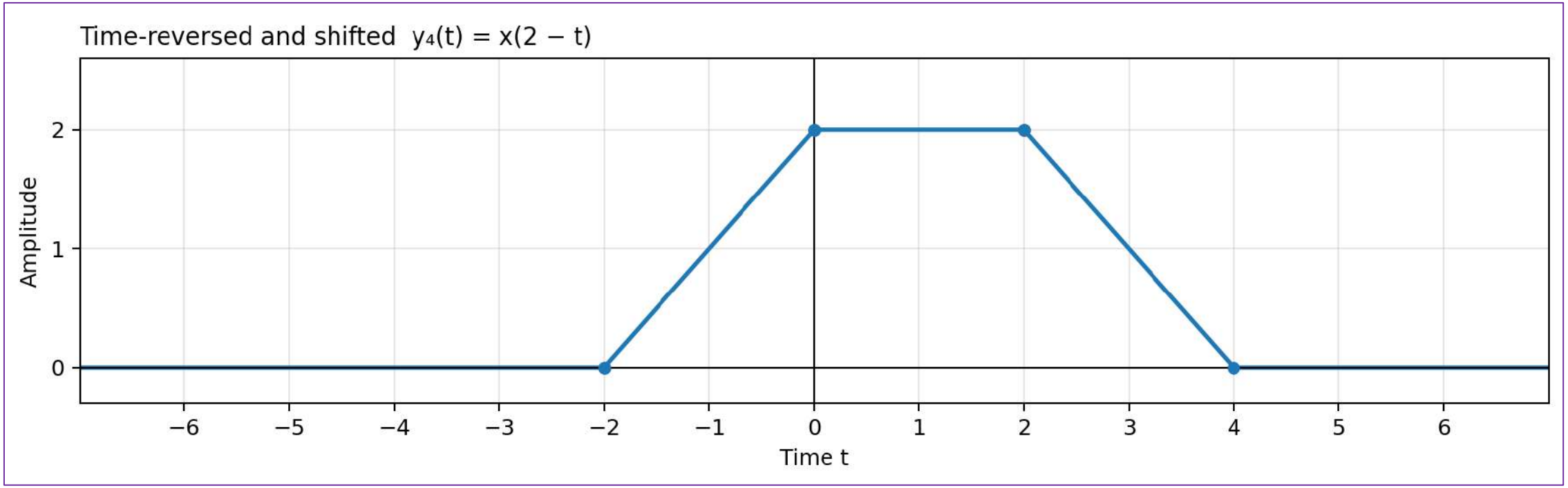
$$y_2(t) = \begin{cases} t+5, & -5 \le t < -3 \\ 2, & -3 \le t < -1 \\ 1-t, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$



$$y_3(t) = \begin{cases} t+4, & -4 \le t < -2 \\ 2, & -2 \le t \le 0 \\ 2-t, & 0 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



$$y_4(t) = \begin{cases} t+2, & -2 \le t < 0 \\ 2, & 0 \le t < 2 \\ 4-t, & 2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$



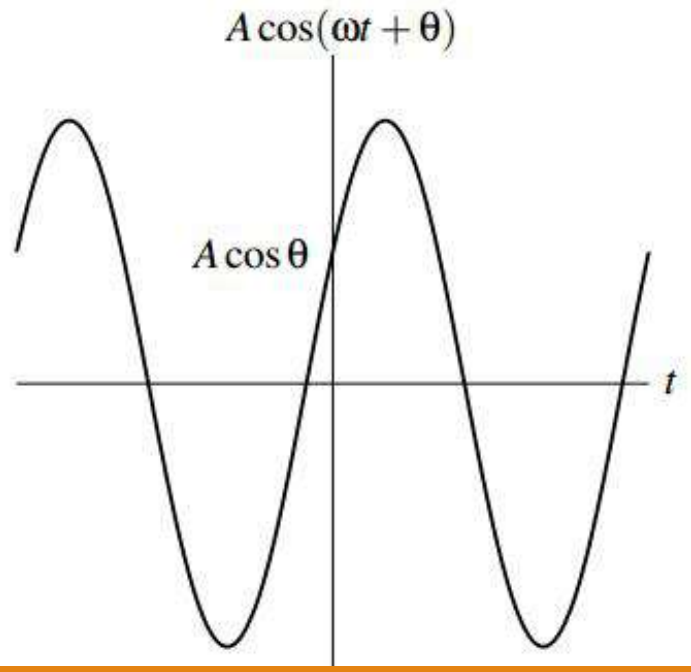
Real Sinusoids

- A (CT) **real sinusoid** is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

where A, ω, and θ are ***real*** constants.

- Such a function is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.
- A real sinusoid has a plot resembling that shown below.



Complex Exponentials

- A (CT) **complex exponential** is a function of the form

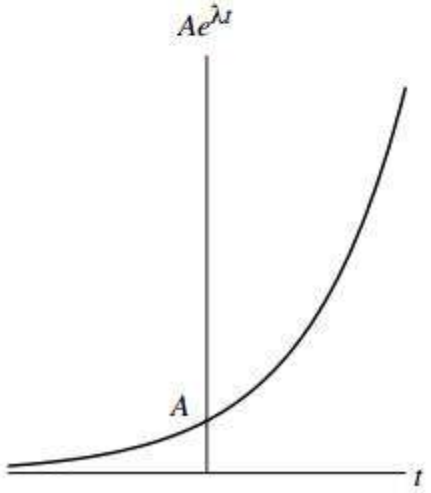
$$x(t) = Ae^{\lambda t},$$

  where A and λ are ***complex*** constants.

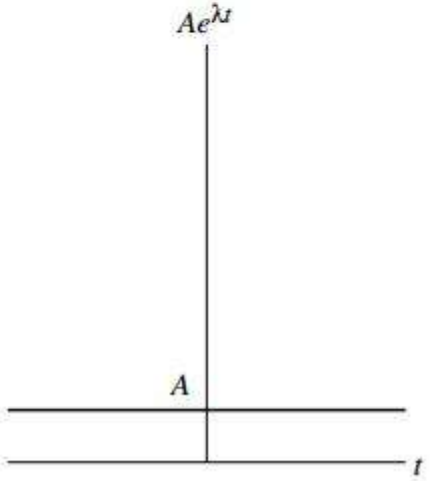
- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

Real Exponentials

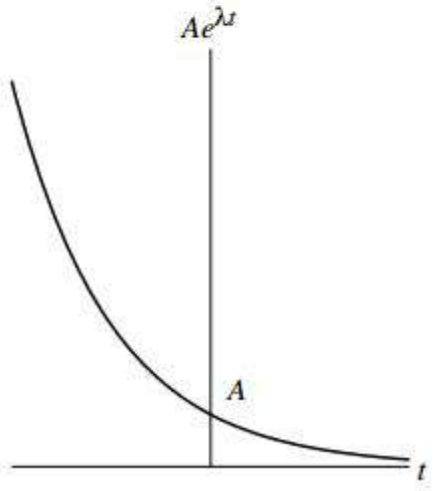
- A **real exponential** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



$\lambda > 0$



$\lambda = 0$



$\lambda < 0$

Complex Sinusoids

- A complex sinusoid is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a (CT) **complex sinusoid** is a function of the form

$$x(t) = Ae^{j\omega t},$$

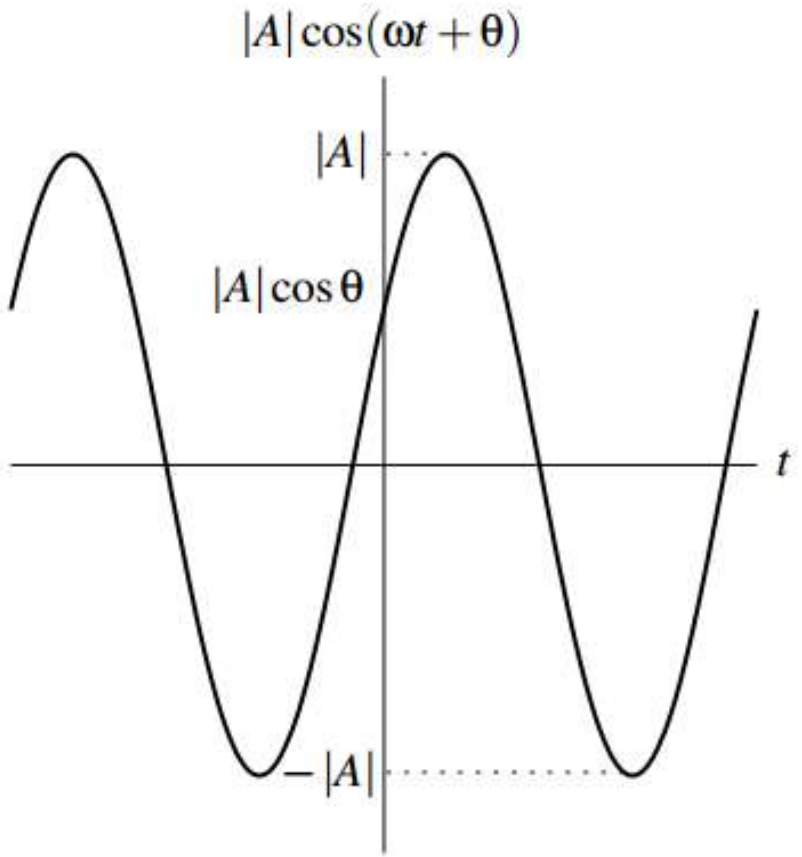
  where A is ***complex*** and ω is ***real***.
- By expressing A in polar form as $A = |A|e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

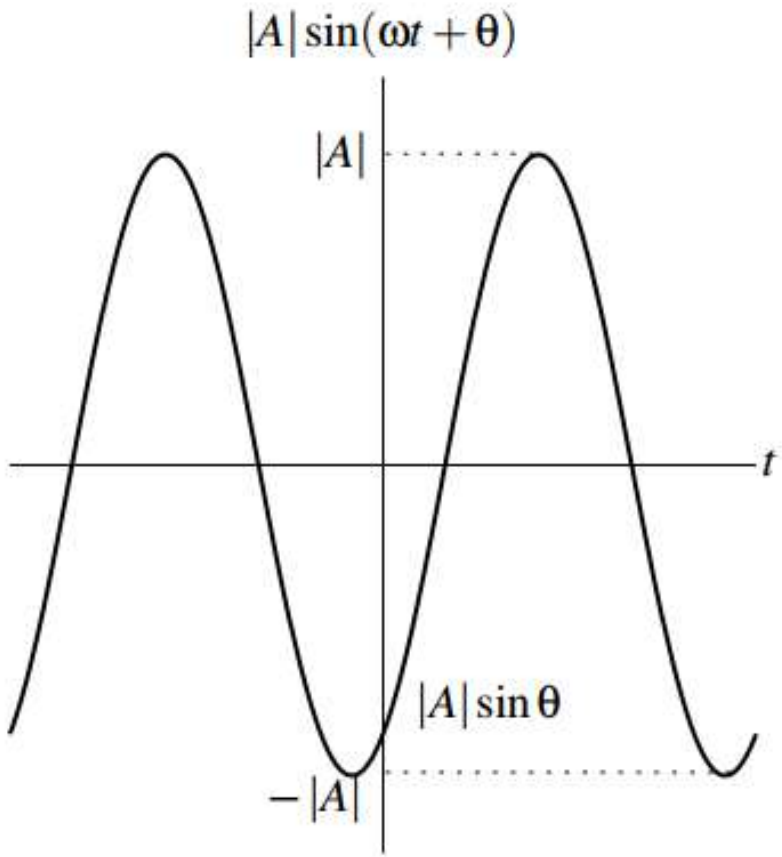
$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.
- Also, x is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

Complex Sinusoids (Continued)

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.





General Complex Exponentials

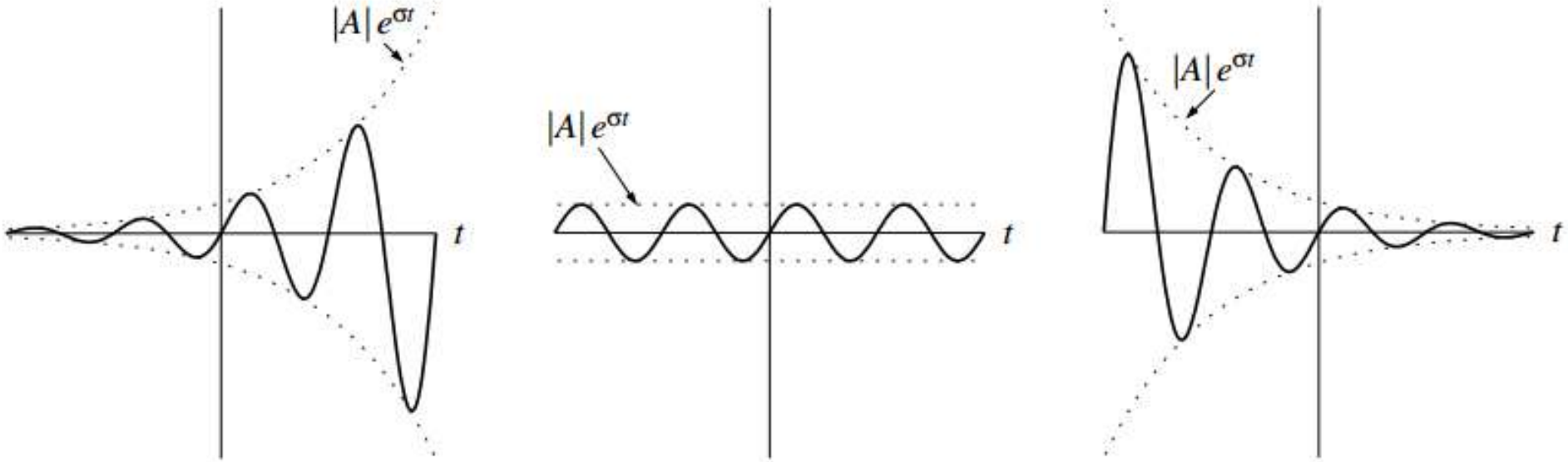
- In the most general case of a complex exponential $x(t) = Ae^{\lambda t}$, A and λ are both ***complex***.
- Letting $A = |A|e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|e^{\sigma t}\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|e^{\sigma t}\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of ***three distinct modes*** of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are ***real sinusoids***.
- If $\sigma > 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $\sigma < 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

General Complex Exponentials (Continued)

- The ***three modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



$\sigma > 0$ $\quad\quad$ $\sigma = 0$ $\quad\quad$ $\sigma < 0$

$$x(t) = 4e^{j10t}$$

Using Euler's formula,

$$x(t) = 4\cos(10t) + j4\sin(10t)$$

$$\mathrm{Re}\{x(t)\} = 4\cos(10t) \qquad \mathrm{Im}\{x(t)\} = 4\sin(10t)$$

Notice that $x(t)$ is a **complex-valued signal**, whereas its real and imaginary parts are ordinary real-valued sinusoidal signals.

Relationship Between Complex Exponentials and Real Sinusoids

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos\omega t + jA\sin\omega t.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$

$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

$$x(t) = 6\cos(100t + 30^\circ)$$

First, convert the phase to radians

$$30^\circ = \frac{\pi}{6}$$

$$x(t) = 6\cos\left(100t + \frac{\pi}{6}\right)$$

Using the cosine identity,

$$x(t) = \frac{6}{2}\left[e^{j(100t+\pi/6)} + e^{-j(100t+\pi/6)}\right]$$

$$x(t) = 3e^{j(100t+\pi/6)} + 3e^{-j(100t+\pi/6)}$$

The original real cosine is represented by two complex exponential components, each with coefficient 3.

$$y(t) = 6\sin\left(100t + \frac{\pi}{6}\right)$$

$$y(t) = \frac{6}{2j}\left[e^{j(100t+\pi/6)} - e^{-j(100t+\pi/6)}\right]$$

$$y(t) = \frac{3}{j}\left[e^{j(100t+\pi/6)} - e^{-j(100t+\pi/6)}\right]$$

Continuous-Time (CT) Systems

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}\{x\},$$

  where $\mathcal{H}$ denotes an operator (i.e., transformation).
- Note that the operator $\mathcal{H}$ ***maps a function to a function*** (not a number to a number).
- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$

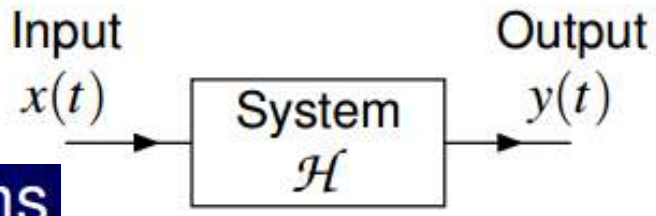
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \to y.$$

- Note that the symbols "$\to$" and "=" have ***very different*** meanings.
- The symbol "$\to$" should be read as "***produces***" (not as "equals").

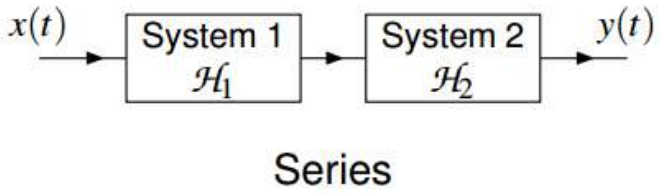
Block Diagram Representations

- Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a ***block diagram*** as shown below.

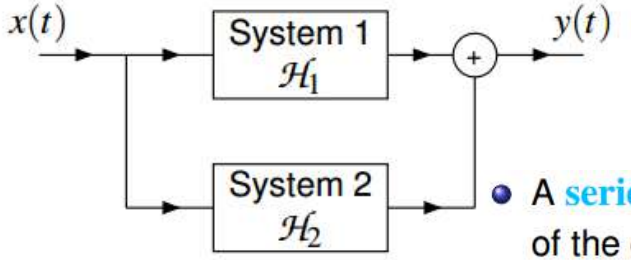


Interconnection of Systems

- ***Two basic ways*** in which systems can be interconnected are shown below.



Series



Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2\{\mathcal{H}_1\{x\}\}.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

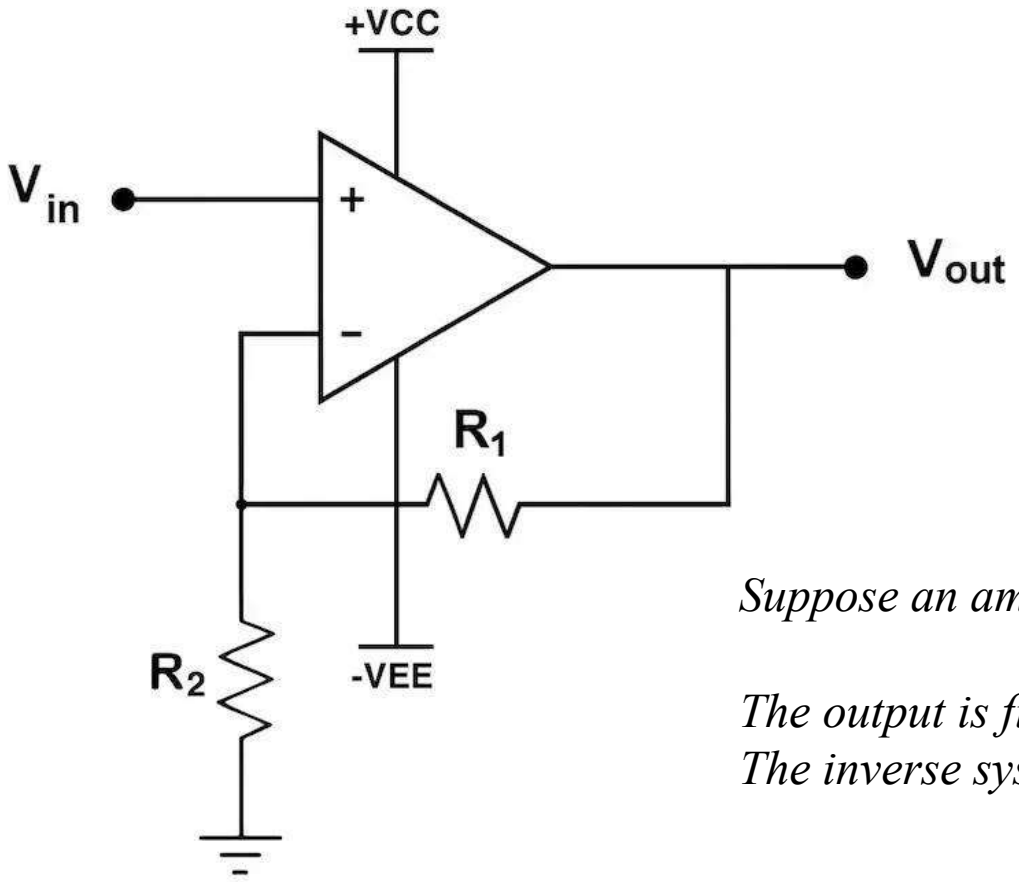
$$y = \mathcal{H}_1\{x\} + \mathcal{H}_2\{x\}.$$

Memory and Causality

- A system with input x and output y is said to have **memory** if, for any real t_0, $y(t_0)$ depends on $x(t)$ for some $t \neq t_0$.
- A system that does not have memory is said to be **memoryless**.
- Although simple, a memoryless system is ***not very flexible***, since its current output value cannot rely on past or future values of the input.
- A system with input x and output y is said to be **causal** if, for every real t_0, $y(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- If the independent variable t represents time, a system must be causal in order to be ***physically realizable***.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable ***need not always represent time***. For example, in some situations, the independent variable might represent position.

Invertibility

- The **inverse** of a system $\mathcal{H}$ is another system $\mathcal{H}^{-1}$ such that the combined effect of $\mathcal{H}$ cascaded with $\mathcal{H}^{-1}$ is a system where the input and output are equal.
- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input x can always be ***uniquely*** determined from its output y.
- Note that the invertibility of a system (which involves mappings between ***functions***) and the invertibility of a function (which involves mappings between ***numbers***) are ***fundamentally different*** things.
- An invertible system will always produce ***distinct outputs*** from any two ***distinct inputs***.
- To show that a system is ***invertible***, we simply find the ***inverse system***.
- To show that a system is ***not invertible***, we find ***two distinct inputs*** that result in ***identical outputs***.
- In practical terms, invertible systems are "nice" in the sense that their ***effects can be undone***.



Suppose an amplifier has a gain of 5.

$$y(t) = 5x(t)$$

The output is five times the input.
The inverse system is:

$$x(t) = \frac{y(t)}{5}$$

Example: If the output voltage is 10 V, the original input voltage is:

$$x(t) = \frac{10}{5} = 2V$$

Bounded-Input Bounded-Output (BIBO) Stability

- A system with input x and output y is **BIBO stable** if, for every bounded x, y is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|y(t)| < \infty$ for all t).
- To show that a system is ***BIBO stable***, we must show that ***every bounded input*** leads to a ***bounded output***.
- To show that a system is ***not BIBO stable***, we only need to find a single ***bounded input*** that leads to an ***unbounded output***.
- In practical terms, a BIBO stable system is ***well behaved*** in the sense that, as long as the system input remains finite for all time, the output will also remain finite for all time.
- Usually, a system that is not BIBO stable will have ***serious safety issues***. For example, an iPod with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized Apple customer and one big lawsuit.

Time Invariance (TI)

- A system $\mathcal{H}$ is said to be **time invariant (TI)** if, for every function x and every real number t_0, the following condition holds:

$$y(t-t_0) = \mathcal{H}x'(t) \quad \text{where} \quad y = \mathcal{H}x \quad \text{and} \quad x'(t) = x(t-t_0)$$

  (i.e., $\mathcal{H}$ ***commutes with time shifts***).

- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an ***identical time shift*** in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior ***does not change*** with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much ***easier to design and analyze***, since their behavior does not change with respect to time.

Additivity, Homogeneity, and Linearity

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

  (i.e., $\mathcal{H}$ ***commutes with sums***).
- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

  (i.e., $\mathcal{H}$ ***commutes with multiplication by a constant***).
- A system that is both additive and homogeneous is said to be **linear**.
- In other words, a system $\mathcal{H}$ is ***linear***, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

  (i.e., $\mathcal{H}$ ***commutes with linear combinations***).
- The linearity property is also referred to as the **superposition** property.
- Practically speaking, linear systems are much ***easier to design and analyze*** than nonlinear systems.

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.