

Worked Practice Problems

Complex Exponentials and System Properties

Mr. Abdullah Numani
CPE2613 Signals and Systems, Fall-2026
Capital University of Science and Technology

Part 1: Complex Exponential Signals

Key facts: complex exponential signals

General complex exponential

$$x(t) = Ce^{st}, \qquad s = \sigma + j\omega$$

Split the exponent

$$x(t) = Ce^{\sigma t}\, e^{j\omega t}$$

Euler's relation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

So, for real C:

$$\mathrm{Re}\{x\} = Ce^{\sigma t}\cos\omega t, \quad \mathrm{Im}\{x\} = Ce^{\sigma t}\sin\omega t$$

Mode of behaviour: look at σ

$\sigma < 0$	decaying
$\sigma = 0$	constant amplitude
$\sigma > 0$	growing

Period of the oscillation: look at ω

$$T = \frac{2\pi}{|\omega|}$$

$Ce^{\sigma t}$ is the **envelope**; ω sets how fast the signal oscillates inside it.

Question

Consider the general complex exponential signal

$$x(t) = 5e^{(-2+j4\pi)t}.$$

(a) **Determine** σ and ω, and **use** Euler's relation to write $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$.

(b) **Compute** $x(0)$, **determine** the period of the oscillation, and state which mode of behaviour (growing, constant amplitude or decaying) $x(t)$ exhibits.

(c) **Sketch** $\mathrm{Re}\{x(t)\}$ for $t \geq 0$, showing its exponential envelope and its value at $t = 0$.

Practice 1 (a): find σ, ω, then use Euler

Step 1. Compare with $Ce^{(\sigma+j\omega)t}$.

$$x(t) = 5e^{(\underbrace{-2}_{\sigma} + j\,\underbrace{4\pi}_{\omega})t} \quad \Longrightarrow \quad C = 5, \quad \sigma = -2, \quad \omega = 4\pi \text{ rad/s}$$

Step 2. Split the exponent into a real part and an imaginary part.

$$x(t) = 5e^{-2t}\, e^{j4\pi t}$$

Step 3. Apply Euler's relation to $e^{j4\pi t}$.

$$x(t) = 5e^{-2t}\big[\cos(4\pi t) + j\sin(4\pi t)\big]$$

Step 4. Read off the real and imaginary parts.

$$\boxed{\mathrm{Re}\{x(t)\} = 5e^{-2t}\cos(4\pi t)} \qquad \boxed{\mathrm{Im}\{x(t)\} = 5e^{-2t}\sin(4\pi t)}$$

Practice 1 (b): value at $t = 0$, period and mode

Step 1. Put $t = 0$. Any exponential equals 1 at $t = 0$:

$$x(0) = 5e^{0} = \boxed{5}$$

Step 2. Period of the oscillation uses ω only.

$$T = \frac{2\pi}{|\omega|} = \frac{2\pi}{4\pi} = \boxed{0.5 \text{ s}}$$

Step 3. Mode of behaviour uses σ only.

$$\sigma = -2 < 0 \quad \Longrightarrow \quad \boxed{\text{decaying}}$$

Careful

$x(t)$ itself is *not* periodic when $\sigma \neq 0$ (its size keeps changing). T is the period of the oscillation *inside* the envelope.

Practice 1 (c): sketch $\mathrm{Re}\{x(t)\} = 5e^{-2t}\cos(4\pi t)$

Step 1. Draw the envelope $\pm 5e^{-2t}$ as dashed curves, starting at ± 5.

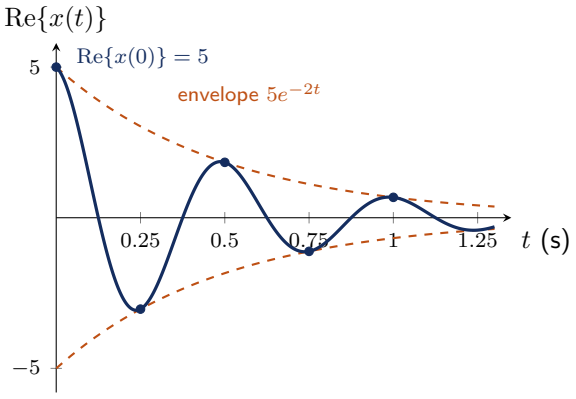
Step 2. Mark one period, $T = 0.5$ s, on the time axis.

Step 3. Start at $\mathrm{Re}\{x(0)\} = 5$ (cosine starts at its peak).

Step 4. Peaks every 0.5 s, troughs halfway between, zeros at $t = 0.125, 0.375, \ldots$

Step 5. The curve touches the envelope at each peak and trough.

t (s)	$\mathrm{Re}\{x(t)\}$
0	5
0.25	$-5e^{-0.5} = -3.03$
0.5	$5e^{-1} = 1.84$
0.75	$-5e^{-1.5} = -1.12$



Practice 2: $x(t) = 4e^{(0.5+j\pi)t}$

Same three parts as Practice 1. Only the numbers change.

(a) σ, ω, real and imaginary parts

Step 1. $C = 4,\ \sigma = 0.5,\ \omega = \pi$ rad/s

Step 2. $x(t) = 4e^{0.5t}\, e^{j\pi t}$

Step 3. $x(t) = 4e^{0.5t}\big[\cos(\pi t) + j\sin(\pi t)\big]$

Step 4. $\mathrm{Re}\{x\} = 4e^{0.5t}\cos(\pi t)$
$\mathrm{Im}\{x\} = 4e^{0.5t}\sin(\pi t)$

(b) $x(0)$, period and mode

Step 1. $x(0) = 4e^{0} = 4$

Step 2. $T = \dfrac{2\pi}{\pi} = 2$ s

Step 3. $\sigma = 0.5 > 0 \Rightarrow$ **growing**

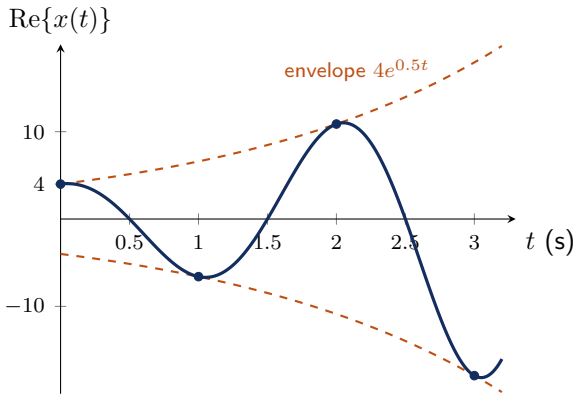
Compare with Practice 1

Only the sign of σ changed the mode: -2 gave decaying, $+0.5$ gives growing.

Practice 2 (c): sketch $\mathrm{Re}\{x(t)\} = 4e^{0.5t}\cos(\pi t)$

Step 1. Envelope $\pm 4e^{0.5t}$, starting at ± 4 and *opening out*.
Step 2. Period $T = 2$ s: peaks at $t = 0, 2$; troughs at $t = 1, 3$.
Step 3. Start at $\mathrm{Re}\{x(0)\} = 4$.
Step 4. Zeros at $t = 0.5, 1.5, 2.5$.
Step 5. Each swing is larger than the last.

t (s)	$\mathrm{Re}\{x(t)\}$
0	4
1	$-4e^{0.5} = -6.59$
2	$4e^{1} = 10.87$
3	$-4e^{1.5} = -17.93$



Part 2: Properties of Systems

The four properties and how to test each one

Property	**Meaning**	**How to test**
Memoryless	$y(t)$ depends only on x at the *same* time t	Look for $x(t \pm a)$, integrals, etc. If none: memoryless
Causal	$y(t)$ never depends on a *future* input $x(t+a)$, $a > 0$	Check every argument of x: is it $\leq t$? (Memoryless $\Rightarrow$ causal)
Linear	Additivity *and* homogeneity hold	Compare $\mathcal{H}\{ax_1 + bx_2\}$ with $a\mathcal{H}\{x_1\} + b\mathcal{H}\{x_2\}$
Time invariant	A delay at the input gives the same delay at the output	Compare $\mathcal{H}\{x(t-t_0)\}$ with $y(t-t_0)$

Quick check for nonlinearity

A linear system must give $y = 0$ when $x = 0$. If $x(t) = 0$ gives $y(t) \neq 0$, the system is **nonlinear**.
(Passing this check does *not* prove linearity; use the full test then.)

Practice 3

Question

Consider the two continuous-time systems

$$\mathcal{H}_1: \; y_1(t) = x(t+3), \qquad\qquad \mathcal{H}_2: \; y_2(t) = 5x(t) - 4.$$

(a) **Apply** the definitions to determine whether $\mathcal{H}_2$ is (i) memoryless or with memory, (ii) causal or noncausal, (iii) linear or nonlinear, and (iv) time invariant or time varying. Explain each answer briefly.

(b) **Sketch** the block diagram of the series connection $\mathcal{H}_1$ followed by $\mathcal{H}_2$, **determine** its overall output $y(t)$ in terms of x, and state whether the overall system is causal and linear.

(c) **Determine** whether $\mathcal{H}_2$ is invertible. If it is, determine its inverse system; if it is not, **show** two distinct inputs that produce the same output.

Practice 3 (a): properties of $\mathcal{H}_2$: $y(t) = 5x(t) - 4$ (1 of 2)

(i) Memory. $y(t)$ uses only $x(t)$, the input at the same instant. No shifts, no integrals.

$$\Rightarrow \textbf{memoryless}$$

(ii) Causality. $y(t)$ never uses a future value $x(t + a)$. (Every memoryless system is causal.)

$$\Rightarrow \textbf{causal}$$

(iii) Linearity, quick check. Put $x(t) = 0$:

$$y(t) = 5(0) - 4 = -4 \neq 0 \quad \Rightarrow \quad \textbf{nonlinear}$$

Why "it is a straight line" is wrong

$y = 5x - 4$ is a straight-line graph, but the constant -4 breaks linearity of the *system*. The full test is on the next slide.

Practice 3 (a): properties of $\mathcal{H}_2$: $y(t) = 5x(t) - 4$ (2 of 2)

(iii) Linearity, full test (homogeneity).

Input $a\,x(t)$:

$$\mathcal{H}_2\{ax\} = 5a\,x(t) - 4$$

a times the original output:

$$a\,\mathcal{H}_2\{x\} = 5a\,x(t) - 4a$$

These differ whenever $a \neq 1$ (e.g. $a = 2$: -4 vs -8).

$\Rightarrow$ **nonlinear**

(iv) Time invariance.

Delay the *input*: $x_2(t) = x(t - t_0)$

$$\mathcal{H}_2\{x_2\} = 5x(t - t_0) - 4$$

Delay the *output*: replace t by $t - t_0$

$$y(t - t_0) = 5x(t - t_0) - 4$$

The two are equal.

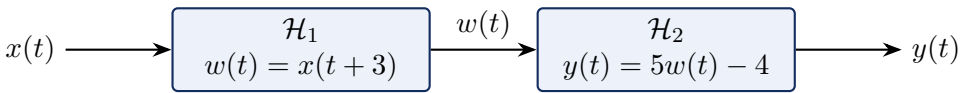
$\Rightarrow$ **time invariant**

Answer for (a)

$\mathcal{H}_2$ is memoryless, causal, nonlinear and time invariant.

Practice 3 (b): series connection, $\mathcal{H}_1$ followed by $\mathcal{H}_2$

Step 1. Draw the block diagram. The output of $\mathcal{H}_1$ (call it w) is the input of $\mathcal{H}_2$.



Step 2. Substitute $w(t)$ into $\mathcal{H}_2$.

$$y(t) = 5\,w(t) - 4 = \boxed{5x(t+3) - 4}$$

Step 3. Causality. At $t = 0$: $y(0) = 5x(3) - 4$ needs the input 3 s in the *future* $\Rightarrow$ **noncausal**.

Step 4. Linearity. $x = 0$ gives $y = -4 \neq 0 \Rightarrow$ **nonlinear**.

Practice 3 (c): is $\mathcal{H}_2: \; y(t) = 5x(t) - 4$ invertible?

Step 1. Can $x(t)$ always be recovered from $y(t)$? Solve the equation for $x(t)$:

$$y(t) = 5x(t) - 4 \quad \Rightarrow \quad 5x(t) = y(t) + 4 \quad \Rightarrow \quad x(t) = \frac{y(t) + 4}{5}$$

There is exactly one $x(t)$ for every $y(t) \Rightarrow$ **invertible**.

Step 2. Write the inverse system (its input is y, its output is x):

$$\boxed{\mathcal{H}_2^{-1}: \quad z(t) = \frac{y(t) + 4}{5}}$$

Step 3. Check by putting $\mathcal{H}_2$ and $\mathcal{H}_2^{-1}$ in series:

$$z(t) = \frac{\big(5x(t) - 4\big) + 4}{5} = x(t) \quad \checkmark$$

Practice 4: parts (a) and (c)

Same three parts as Practice 3. Only the systems change:

$$\mathcal{H}_1:\ y_1(t) = x(t-4), \qquad \mathcal{H}_2:\ y_2(t) = 2 - 3x(t)$$

(a) Properties of $\mathcal{H}_2$

(i) Uses only $x(t) \Rightarrow$ **memoryless**

(ii) No future values $\Rightarrow$ **causal**

(iii) $x = 0$ gives $y = 2 \neq 0$; also
$\mathcal{H}_2\{ax\} = 2 - 3ax$
$a\mathcal{H}_2\{x\} = 2a - 3ax$
not equal $\Rightarrow$ **nonlinear**

(iv) $\mathcal{H}_2\{x(t-t_0)\} = 2 - 3x(t-t_0)$
$y(t-t_0) = 2 - 3x(t-t_0)$
equal $\Rightarrow$ **time invariant**

(c) Invertibility of $\mathcal{H}_2$

Step 1. $y = 2 - 3x \Rightarrow 3x = 2 - y$

Step 2. $\mathcal{H}_2^{-1}:\ z(t) = \dfrac{2 - y(t)}{3}$

Step 3. Check: $\dfrac{2 - (2 - 3x)}{3} = x$ ✓

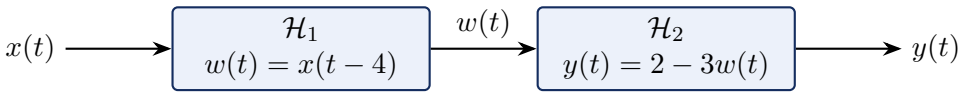
$\Rightarrow$ **invertible**

Same answer pattern as Practice 3

Any $y = ax(t) + b$ with $a \neq 0$, $b \neq 0$ is memoryless, causal, nonlinear, TI and invertible.

Practice 4 (b): series connection, $\mathcal{H}_1$ followed by $\mathcal{H}_2$

Step 1. Draw the block diagram.



Step 2. Substitute $w(t)$ into $\mathcal{H}_2$.

$$y(t) = 2 - 3\,w(t) = \boxed{2 - 3x(t-4)}$$

Step 3. Causality. $y(t)$ uses $x(t-4)$, an input 4 s in the *past* $\Rightarrow$ **causal**.

Step 4. Linearity. $x = 0$ gives $y = 2 \neq 0 \Rightarrow$ **nonlinear**.

Compare with Practice 3

A *delay* $x(t-4)$ keeps the series system causal; an *advance* $x(t+3)$ makes it noncausal.

If the system is *not* invertible: the other branch of part (c)

Example: $\mathcal{H}: \; y(t) = x^2(t) + 1$

Step 1. Try to solve for x. $x^2(t) = y(t) - 1 \Rightarrow x(t) = \pm\sqrt{y(t) - 1}$
Two possible answers: the sign of x is lost.

Step 2. Show two distinct inputs with the same output.

$$x_1(t) = 2 \;\Rightarrow\; y_1(t) = 4 + 1 = 5, \qquad x_2(t) = -2 \;\Rightarrow\; y_2(t) = 4 + 1 = 5$$

Step 3. Conclude. $x_1 \neq x_2$ but $y_1 = y_2$, so the input cannot be recovered from the output $\Rightarrow$ **not invertible**.

In your answer

One clear pair of inputs is enough. Write both inputs, both outputs, and the conclusion.

Summary

Common mistakes to avoid

Complex exponentials

- Using σ for the period. The period uses ω: $T = 2\pi/|\omega|$.
- Writing $T = 2\pi\omega$ instead of $2\pi/\omega$.
- Using ω for the mode. The mode depends only on the sign of σ.
- Drawing the cosine without the envelope, or not touching it at the peaks.
- Forgetting to label the value at $t = 0$.

System properties

- Calling $y = ax + b$ linear because its graph is a straight line.
- In the TI test, shifting the t outside $x(\cdot)$ when you delay the input.
- Judging causality of the series system from $\mathcal{H}_2$ alone. The shift in $\mathcal{H}_1$ decides it.
- Giving the inverse without checking it.
- Writing only "yes/no" with no reason.

Step-by-step checklist

Complex exponentials

1. Read C, σ, ω from the exponent.
2. Split: $Ce^{\sigma t}e^{j\omega t}$, apply Euler.
3. $x(0) = C$; $T = 2\pi/|\omega|$; mode from the sign of σ.
4. Sketch: envelope first, then the cosine from $x(0)$.

System properties

1. Memory, causality: look at the arguments of x.
2. Linearity: zero-input check, then homogeneity.
3. Time invariance: delayed input vs. delayed output.
4. Series: substitute $\mathcal{H}_1$ into $\mathcal{H}_2$.
5. Inverse: solve for x, then check.

Remember

Every answer needs a one-line reason from the definition. A bare yes/no gets no credit.