

Signals and Systems: Worked Problems

Signal Classification, System Properties and Convolution

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Roadmap: four problems, eight parts

Problem	Part (a)	Part (b)
1 Signals	Even, odd or neither (CT and DT)	Energy or power signal; find E or P
2 Systems	Five properties of four systems	Invertible or not; inverse or counterexample
3 Convolution	$y(t) = x(t) * h(t)$ by flip, shift, integrate	Sketch $y(t)$ and read off one value
4 LTI tools	$h(t)$ from $s(t)$, then a pulse response	Series and parallel connections; causality and stability

How to use these slides

Each problem starts with the key facts you need, then a worked solution broken into numbered steps. Try each problem yourself before moving to the solution slides.

Problem 1: Classifying Signals

Key facts: even and odd signals

Definitions

Even: $x(-t) = x(t)$ Odd: $x(-t) = -x(t)$

(Same for discrete time with n in place of t.)

Common building blocks

Even	Odd
$t^2, t^4, \ldots$	$t, t^3, t^5, \ldots$
$\cos(\omega t)$	$\sin(\omega t)$
$\lvert t \rvert$, constants	$\operatorname{sgn}(t)$

Product rules

even $\times$ even $=$ **even**
odd $\times$ odd $=$ **even**
even $\times$ odd $=$ **odd**

Always confirm

Use the rules to predict, then *substitute* $-t$ (or $-n$) and simplify to confirm. A sum such as $t + \cos t$ is **neither**.

Problem 1

Problem

(a) Decide whether each signal is even, odd, or neither. Support each decision with the symmetry of its factors and a direct substitution.

(i) $x(t) = t^5 \cos(3t)$ (ii) $x[n] = n^3 \sin\left(\frac{\pi n}{6}\right)$

(b) Decide whether each signal has finite energy or finite average power, and find the value of that quantity.

(i) $x(t) = 5e^{-3t}u(t)$ (ii) $x(t) = 6\cos(4t)$

Problem 1 (a)(i): $x(t) = t^5 \cos(3t)$

Step 1. Classify each factor.

t^5: $(-t)^5 = -t^5 \Rightarrow$ odd

$\cos(3t)$: $\cos(-3t) = \cos(3t) \Rightarrow$ even

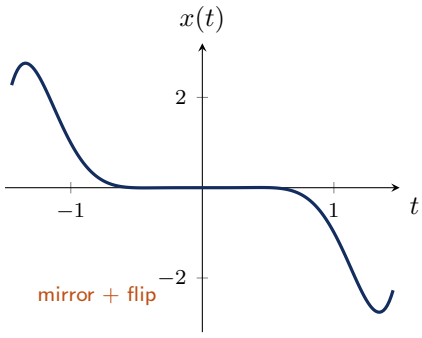
Step 2. Apply the product rule.

odd $\times$ even $=$ odd

Step 3. Confirm by substitution.

$$x(-t) = (-t)^5 \cos(-3t) = -t^5 \cos(3t) = -x(t)$$

$$\Longrightarrow \quad \boxed{\text{Odd}}$$



The graph is symmetric about the **origin**: the left half is the right half rotated by $180°$.

Problem 1 (a)(ii): $x[n] = n^3 \sin\left(\frac{\pi n}{6}\right)$

Step 1. Classify each factor.

n^3: $(-n)^3 = -n^3 \Rightarrow$ odd

$\sin\left(\frac{\pi n}{6}\right)$: $\sin(-\theta) = -\sin\theta \Rightarrow$ odd

Step 2. Apply the product rule.

odd $\times$ odd $=$ even (the two minus signs cancel)

Step 3. Confirm by substitution.

$$x[-n] = (-n^3)\left(-\sin\tfrac{\pi n}{6}\right) = n^3 \sin\tfrac{\pi n}{6} = x[n]$$

$$\Longrightarrow \quad \boxed{\text{Even}}$$

Numerical check

n	$x[n]$	$x[-n]$
1	0.5	$(-1)(-0.5) = 0.5$
2	6.93	$(-8)(-0.866) = 6.93$
3	27	$(-27)(-1) = 27$

The same product rules hold in discrete time.

Key facts: energy and power signals

Energy

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Average power

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

For a periodic signal with period T_0:

$$P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 \, dt$$

Classification

Energy signal	$0 < E < \infty, \quad P = 0$
Power signal	$0 < P < \infty, \quad E = \infty$

Rule of thumb

Signals that **die out** (e.g. decaying exponentials) are energy signals. Signals that **go on for ever** at a steady size (e.g. sinusoids) are power signals.

Useful result: $A\cos(\omega_0 t + \phi)$ has $P = A^2/2$.

Problem 1 (b)(i): $x(t) = 5e^{-3t}u(t)$

Step 1. Square the signal. $u(t)$ limits it to $t \geq 0$:

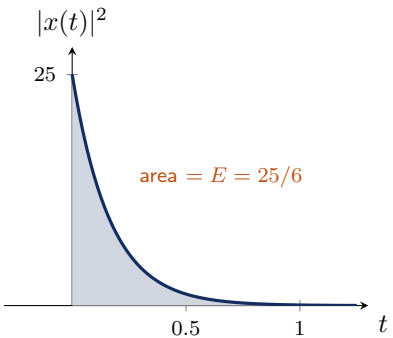
$$|x(t)|^2 = 25e^{-6t}, \quad t \geq 0$$

Step 2. Integrate to find the energy.

$$E = \int_0^\infty 25e^{-6t}\,dt = 25\left[\frac{e^{-6t}}{-6}\right]_0^\infty = 25 \cdot \frac{1}{6}$$

Step 3. Conclude. E is finite and nonzero, so $P = 0$.

Energy signal, $E = \frac{25}{6} \approx 4.17$



General result: $Ae^{-at}u(t)$ with $a > 0$ has $E = A^2/(2a)$.

Problem 1 (b)(ii): $x(t) = 6\cos(4t)$

Step 1. Is the energy finite? The signal is periodic with $T_0 = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ and never decays. Each period adds the same positive energy, so over infinitely many periods $E = \infty$.

Step 2. Average power over one period. Use $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$:

$$P = \frac{1}{T_0}\int_0^{T_0} 36\cos^2(4t)\,dt = \frac{36}{T_0}\int_0^{T_0} \frac{1+\cos(8t)}{2}\,dt = \frac{36}{T_0}\left(\frac{T_0}{2} + 0\right) = 18$$

(The $\cos(8t)$ term completes whole cycles in T_0, so it integrates to zero.)

Step 3. Conclude.

Power signal, $P = 18,\ E = \infty$

Check: $P = \dfrac{A^2}{2} = \dfrac{6^2}{2} = 18$ ✓

Problem 2: Properties of Systems

The five properties and how to check each one

Property	Meaning	How to check
Memoryless	$y(t)$ depends only on x at the *same* instant t	Look for shifted or scaled arguments, integrals. None: memoryless
Causal	$y(t)$ never uses a *future* input	Every argument of x must be $\leq t$ for *all* t, including $t < 0$
Linear	Additivity *and* homogeneity	Compare $\mathcal{H}\{ax_1 + bx_2\}$ with $a\mathcal{H}\{x_1\} + b\mathcal{H}\{x_2\}$
Time invariant	Delaying the input only delays the output	Compare $\mathcal{H}\{x(t - t_0)\}$ with $y(t - t_0)$
BIBO stable	Bounded input gives bounded output	Assume $\lvert x\rvert \leq B$ and bound $\lvert y\rvert$; or find one bounded input whose output blows up

Two shortcuts

Memoryless $\Rightarrow$ causal. If $x = 0$ gives $y \neq 0$, the system is nonlinear.

Problem

(a) For each continuous-time system below, decide whether it is memoryless, causal, linear, time invariant and BIBO stable. Give a one-line reason for each decision.

$$\text{(i) } y(t) = 4x(t) + 5 \qquad \text{(ii) } y(t) = x\left(\tfrac{t}{3}\right)$$

$$\text{(iii) } y(t) = t^2 x(t) \qquad \text{(iv) } y(t) = 2\int_{-\infty}^{t} x(\tau)\, d\tau$$

(b) Decide which of these two systems can be undone. For the one that can, find the system that recovers $x(t)$; for the one that cannot, give two different inputs with the same output.

$$\text{(i) } y(t) = 4x(t) + 5 \qquad \text{(ii) } y(t) = \sin\big(x(t)\big)$$

Problem 2 (a)(i): $y(t) = 4x(t) + 5$

Memory. Uses only $x(t)$ at the same instant $\Rightarrow$ **memoryless**.

Causality. No future values (memoryless $\Rightarrow$ causal) $\Rightarrow$ **causal**.

Linearity. Zero input: $x = 0 \Rightarrow y = 5 \neq 0$. Full check (additivity):

$$\mathcal{H}\{x_1 + x_2\} = 4(x_1 + x_2) + 5 \qquad \text{but} \qquad y_1 + y_2 = 4(x_1 + x_2) + 10 \quad \Rightarrow \quad \textbf{nonlinear}$$

Time invariance. Delayed input: $4x(t - t_0) + 5$. Delayed output: $y(t - t_0) = 4x(t - t_0) + 5$. Equal $\Rightarrow$ **time invariant**.

Stability. $|x(t)| \leq B \;\Rightarrow\; |y(t)| \leq 4B + 5 < \infty \Rightarrow$ **stable**.

Common trap

$y = 4x + 5$ has a straight-line graph, but the constant $+5$ makes the *system* nonlinear.

Problem 2 (a)(ii): $y(t) = x(t/3)$ (time scaling)

Memory. $y(3) = x(1)$: the output at $t = 3$ uses a different instant $\Rightarrow$ **with memory**.

Causality. Check negative time too: $y(-3) = x(-1)$, and $-1 > -3$, so the output at $t = -3$ needs an input from the *future* $\Rightarrow$ **noncausal**.

Linearity. $\mathcal{H}\{ax_1 + bx_2\} = ax_1(t/3) + bx_2(t/3) = ay_1(t) + by_2(t) \Rightarrow$ **linear**.

Time invariance. Let $x_1(t) = x(t - t_0)$. Then

$$\mathcal{H}\{x_1\} = x_1(t/3) = x\big(\tfrac{t}{3} - t_0\big) \qquad \text{but} \qquad y(t - t_0) = x\big(\tfrac{t-t_0}{3}\big) = x\big(\tfrac{t}{3} - \tfrac{t_0}{3}\big)$$

Not equal $\Rightarrow$ **time varying**.

Stability. $|y(t)| = |x(t/3)| \leq B \Rightarrow$ **stable**.

Problem 2 (a)(iii): $y(t) = t^2 x(t)$ (time-dependent gain)

Memory. Uses only $x(t)$; the factor t^2 is a gain that depends on time $\Rightarrow$ **memoryless**.

Causality. No future values $\Rightarrow$ **causal**.

Linearity. $t^2\big[ax_1(t) + bx_2(t)\big] = a\,t^2 x_1(t) + b\,t^2 x_2(t) \Rightarrow$ **linear**.

Time invariance. Delayed input gives $t^2 x(t - t_0)$. Delayed output gives $(t - t_0)^2 x(t - t_0)$. Not equal $\Rightarrow$ **time varying**.

Stability. The bounded input $x(t) = 1$ gives $y(t) = t^2$, which grows without bound $\Rightarrow$ **unstable**.

Pattern

Whenever t appears *outside* the argument of $x(\cdot)$, the system is time varying.

Problem 2 (a)(iv): $y(t) = 2\int_{-\infty}^{t} x(\tau)\,d\tau$ (scaled integrator)

Memory. The output adds up all past input values $\Rightarrow$ **with memory**.

Causality. The upper limit is t, so only $\tau \le t$ is used $\Rightarrow$ **causal**.

Linearity. The integral of $ax_1 + bx_2$ splits into a and b times separate integrals $\Rightarrow$ **linear**.

Time invariance. Delayed input, then substitute $\lambda = \tau - t_0$:

$$2\int_{-\infty}^{t} x(\tau - t_0)\,d\tau = 2\int_{-\infty}^{t-t_0} x(\lambda)\,d\lambda = y(t - t_0) \quad \Rightarrow \quad \textbf{time invariant}$$

Stability. The bounded input $x(t) = u(t)$ gives $y(t) = 2t\,u(t)$, a ramp that grows without bound $\Rightarrow$ **unstable**.

Problem 2 (a): summary

System	Memoryless	Causal	Linear	Time invariant	BIBO stable
(i) $4x(t) + 5$	**Yes**	**Yes**	**No**	**Yes**	**Yes**
(ii) $x(t/3)$	**No**	**No**	**Yes**	**No**	**Yes**
(iii) $t^2 x(t)$	**Yes**	**Yes**	**Yes**	**No**	**No**
(iv) $2\int_{-\infty}^{t} x(\tau)\, d\tau$	**No**	**Yes**	**Yes**	**Yes**	**No**

Remember

A bare "Yes/No" explains nothing. Each entry needs its one-line reason or counterexample.

Problem 2 (b)(i): undoing $y(t) = 4x(t) + 5$

Idea. A system is *invertible* when different inputs always give different outputs, so $x(t)$ can be recovered uniquely from $y(t)$.

Step 1. Solve for $x(t)$.

$$y(t) = 4x(t) + 5 \;\Rightarrow\; 4x(t) = y(t) - 5 \;\Rightarrow\; x(t) = \frac{y(t) - 5}{4}$$

Exactly one $x(t)$ for every $y(t)$ $\Rightarrow$ **invertible**.

Step 2. Write the inverse system (input w, output z):

$$\boxed{\mathcal{H}^{-1}:\; z(t) = \tfrac{1}{4}\, w(t) - \tfrac{5}{4}}$$

Step 3. Check by placing $\mathcal{H}$ and $\mathcal{H}^{-1}$ in series:

$$\tfrac{1}{4}\big(4x(t) + 5\big) - \tfrac{5}{4} = x(t) \;\checkmark$$

Problem 2 (b)(ii): $y(t) = \sin\big(x(t)\big)$ cannot be undone

Step 1. Why it fails. Sine is periodic, so a horizontal line meets its graph many times: one output, many possible inputs.

Step 2. Two different inputs, same output.

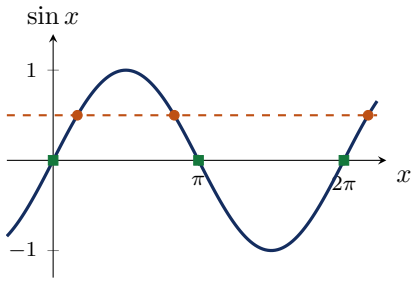
$$x_1(t) = 0 \;\Rightarrow\; y = \sin 0 = 0$$

$$x_2(t) = \pi \;\Rightarrow\; y = \sin \pi = 0$$

Step 3. Conclude. $x_1 \neq x_2$ but $y_1 = y_2$:

Not invertible

Another pair: x and $\pi - x$ always give the same output.



Each horizontal line hits several input values.

Problem 3: Convolution of Continuous-Time Signals

Key facts: the convolution integral

For an LTI system with impulse response $h(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Six steps

1. Draw $x(\tau)$ and $h(\tau)$ against τ.
2. **Flip** h: $h(-\tau)$.
3. **Shift** by t: $h(t-\tau)$.
4. Find the **intervals of** t where the overlap changes form.
5. In each interval, **multiply** and **integrate** over the overlap.
6. Collect the pieces; check continuity.

Finding the limits

If x lives on $[a, \infty)$ and $h(t-\tau)$ lives on $[t-L,\, t]$, the overlap is

$$\max(a,\, t-L) \;\le\; \tau \;\le\; t$$

(empty when $t < a$).

Useful checks

- $y(t)$ is continuous if neither signal has impulses.
- Area of y = (area of x) $\times$ (area of h).

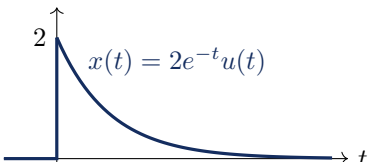
Problem 3

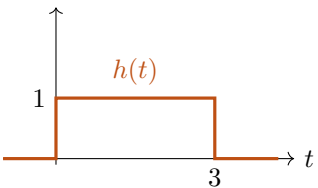
Problem

An LTI system has impulse response $h(t) = u(t) - u(t-3)$ (a unit-height pulse on $0 \le t < 3$). Its input is $x(t) = 2e^{-t}u(t)$.

(a) Evaluate $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ for every t. Draw $x(\tau)$ and $h(t-\tau)$ for each range of t you use.

(b) Plot $y(t)$ and find its value at $t = 3$.





Problem 3 (a), Steps 1–3: flip and shift the pulse

Step 1. Keep the input fixed: $x(\tau) = 2e^{-\tau}u(\tau)$.

Step 2. Flip. $h(-\tau)$ is the pulse reflected about $\tau = 0$: it occupies $-3 \le \tau \le 0$.

Step 3. Shift by t.

$$h(t-\tau) = \begin{cases} 1, & t-3 \le \tau \le t, \\ 0, & \text{otherwise.} \end{cases}$$

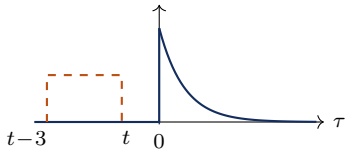
This is a unit-height **window** from $\tau = t-3$ to $\tau = t$ that slides to the right as t increases.

Consequence. Inside the window $h = 1$, and $x(\tau) = 0$ for $\tau < 0$, so

$$y(t) = \int_{\max(0,\, t-3)}^{t} 2e^{-\tau}\, d\tau \qquad \text{(whenever the window reaches } \tau \ge 0\text{)}.$$

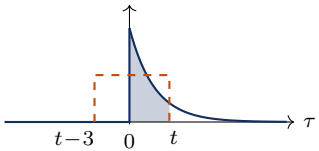
Problem 3 (a), Step 4: the three ranges of t

The window edges $t-3$ and t cross $\tau = 0$ at $t = 0$ and at $t = 3$. That gives three cases.



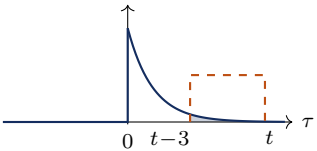
Case 1: $t < 0$

no overlap



Case 2: $0 \le t < 3$

overlap $0 \le \tau \le t$



Case 3: $t \ge 3$

overlap $t-3 \le \tau \le t$

$x(\tau) = 2e^{-\tau}u(\tau)$ — $h(t-\tau)$, height 1 — overlap (area $= y(t)$)

Problem 3 (a), Step 5: integrate in each case

Case 1: $t < 0$. The window lies entirely in $\tau < 0$, where $x(\tau) = 0$:

$$y(t) = 0$$

Case 2: $0 \le t < 3$. The right edge has passed $\tau = 0$, the left edge has not:

$$y(t) = \int_0^t 2e^{-\tau}\, d\tau = 2\Big[-e^{-\tau} \Big]_0^t = 2\left(1 - e^{-t}\right)$$

Case 3: $t \ge 3$. The whole window lies in $\tau \ge 0$:

$$y(t) = \int_{t-3}^t 2e^{-\tau}\, d\tau = 2\left(e^{-(t-3)} - e^{-t}\right) = 2\left(e^3 - 1\right) e^{-t}$$

Problem 3 (a), Step 6: collect and check

$$y(t) = \begin{cases} 0, & t < 0, \\ 2\left(1 - e^{-t}\right), & 0 \le t < 3, \\ 2\left(e^{3} - 1\right) e^{-t}, & t \ge 3. \end{cases}$$

Continuity check

$t = 0$: both sides give 0 ✓

$t = 3$: Case 2 gives $2(1 - e^{-3})$;
Case 3 gives $2(e^3 - 1)e^{-3} = 2(1 - e^{-3})$ ✓

Area check

Area of $x = 2$, area of $h = 3$, so area of y should be 6:

$\int_0^3 2(1 - e^{-t})\, dt = 6 - 2(1 - e^{-3})$
$\int_3^\infty 2(e^3 - 1)e^{-t}\, dt = 2(1 - e^{-3})$
Sum $= 6$ ✓

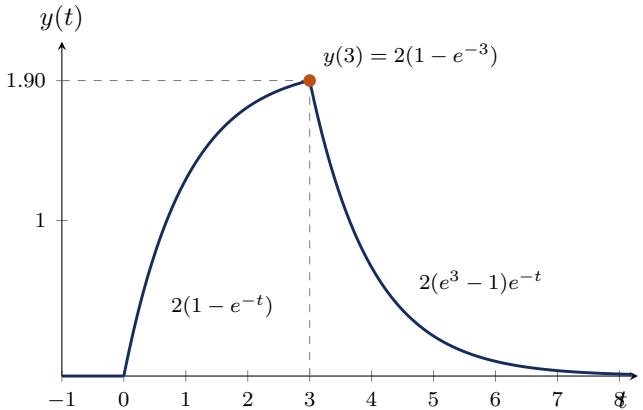
Problem 3 (b): plot of $y(t)$ and the value at $t = 3$

Value at $t = 3$.

$$y(3) = 2\left(1 - e^{-3}\right) \approx \boxed{1.90}$$

Shape.

- $t < 0$: zero.
- $0 \le t < 3$: rises like a charging curve while the window fills.
- Peak at $t = 3$, the pulse length.
- $t > 3$: decays like e^{-t} as the window slides past.



Problem 4: Step Response and Interconnected Systems

Key facts: tools for LTI systems

Step and impulse response

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \quad \Longleftrightarrow \quad h(t) = \frac{d}{dt}\, s(t)$$

Using linearity and time invariance

If $u(t) \to s(t)$, then $u(t - t_0) \to s(t - t_0)$, so

$$u(t) - u(t - t_0) \;\to\; s(t) - s(t - t_0)$$

Shift property

$$f(t) * \delta(t - t_0) = f(t - t_0)$$

Interconnections

Series: $h_s = h_1 * h_2$ (order does not matter)

Parallel: $h_p = h_1 + h_2$

Properties read from $h(t)$

Causal $\Leftrightarrow$ $h(t) = 0$ for $t < 0$

BIBO stable $\Leftrightarrow$ $\displaystyle\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

Problem 4

Problem

(a) An LTI system responds to the unit step with $s(t) = 2\left(1 - e^{-2t}\right) u(t)$. Find its impulse response $h(t)$. Then, without evaluating a convolution integral, find its response to the pulse $x(t) = u(t) - u(t-1)$.

(b) Two LTI systems have $h_1(t) = \delta(t-2)$ and $h_2(t) = e^{-3t}u(t)$. Draw them connected in series and in parallel, and find the overall impulse response in each case. Is the series combination causal? Is it BIBO stable?

Problem 4 (a), part 1: impulse response from the step response

Step 1. Why differentiate? $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$, so the step response is the running integral of h, and $h(t) = \dfrac{d}{dt} s(t)$.

Step 2. Product rule, with $\dfrac{d}{dt} u(t) = \delta(t)$:

$$h(t) = \frac{d}{dt}\left[2\left(1 - e^{-2t}\right) u(t)\right] = 4e^{-2t}u(t) + \underbrace{2\left(1 - e^{-2t}\right)\delta(t)}_{=2(1-e^{0})\,\delta(t)\,=\,0}$$

Step 3. The impulse term vanishes because $f(t)\delta(t) = f(0)\delta(t)$ and $f(0) = 0$ here.

$$\boxed{h(t) = 4e^{-2t}u(t)}$$

Check. $\displaystyle\int_0^t 4e^{-2\tau}\, d\tau = 2\left(1 - e^{-2t}\right) = s(t)$ for $t \geq 0$ ✓

Problem 4 (a), part 2: response to the pulse $u(t) - u(t-1)$

Step 1. Linearity and time invariance.

$$u(t) \to s(t), \qquad u(t-1) \to s(t-1)$$

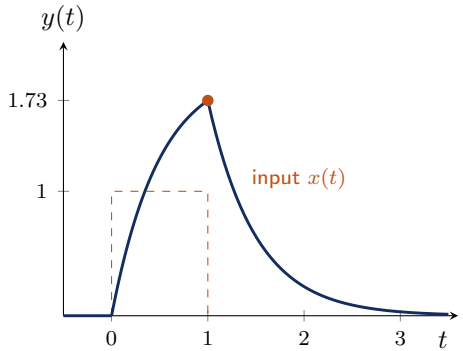
Step 2. Subtract. $\boxed{y(t) = s(t) - s(t-1)}$

$$= 2\left(1 - e^{-2t}\right) u(t) - 2\left(1 - e^{-2(t-1)}\right) u(t-1)$$

Step 3. Piecewise form.

$$y(t) = \begin{cases} 0, & t < 0, \\ 2\left(1 - e^{-2t}\right), & 0 \le t < 1, \\ 2\left(e^{2} - 1\right) e^{-2t}, & t \ge 1. \end{cases}$$

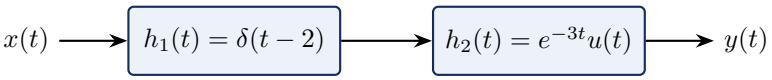
Peak: $y(1) = 2(1 - e^{-2}) \approx 1.73$.



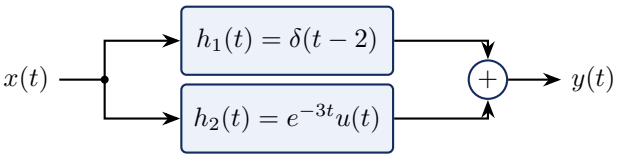
Rises while the pulse is on, then decays once it switches off.

Problem 4 (b): block diagrams

Series



Parallel



Series: the output of the first block is the input of the second. Parallel: both blocks see the same input and their outputs are added.

Problem 4 (b): overall impulse responses, causality and stability

Series. Convolve. Convolving with $\delta(t-2)$ just delays by 2:

$$h_s(t) = \delta(t-2) * e^{-3t}u(t) \quad \Longrightarrow \quad \boxed{h_s(t) = e^{-3(t-2)}u(t-2)}$$

Parallel. Add:

$$\boxed{h_p(t) = \delta(t-2) + e^{-3t}u(t)}$$

Series system: causal?

$h_s(t) = 0$ for $t < 2$, so certainly for $t < 0$ $\Rightarrow$ **causal**. (A 2-second delay followed by a causal first-order system.)

Series system: BIBO stable?

$$\int_2^\infty e^{-3(t-2)}\,dt = \left[\frac{e^{-3(t-2)}}{-3}\right]_2^\infty = \frac{1}{3} < \infty$$

$\Rightarrow$ **BIBO stable**.

Summary

Common mistakes to avoid

Signals and systems

- Stating even/odd from the rules without substituting $-t$.
- Forgetting $u(t)$ when setting the energy limits.
- Saying a sinusoid has finite energy.
- Calling $y = ax + b$ linear because its graph is a line.
- Checking causality of $x(t/a)$ only for $t > 0$.
- Inverting a system without checking the result.

Convolution and LTI tools

- Flipping x and h both, or forgetting to flip.
- Wrong limits: the lower limit is $\max(0, t - L)$.
- Missing a range of t (e.g. $t < 0$).
- Leaving out the $\delta(t)$ term without showing it is zero.
- Adding impulse responses for a series connection (it is convolution).

Step-by-step checklist

Problems 1 and 2

1. Even/odd: classify factors, product rule, substitute.
2. Energy/power: square, integrate, or average over one period.
3. Each system: memory, causality, linearity, TI, stability, one reason each.
4. Invertible: solve for x and check, or give two inputs with one output.

Problems 3 and 4

1. Convolution: flip, shift, list the ranges of t, integrate each, check.
2. Plot: mark the peak and its value.
3. $h = ds/dt$; pulse response $= s(t) - s(t - t_0)$.
4. Series: convolve. Parallel: add.
5. Causal: $h = 0$ for $t < 0$. Stable: $\int |h| < \infty$.

Remember

Show every step. A correct final answer with no working shows little understanding.